THE INTEGRALS OF MECHANICS

BY

OLIVER CLARENCE LESTER, PH.D.

PROFESSOR OF PHYSICS, UNIVERSITY OF COLORADO; FORMERLY
INSTRUCTOR IN PHYSICS IN THE SHEFFIELD SCIEN-
TIFIC SCHOOL, YALE UNIVERSITY

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PREFACE

The matter presented in the following pages was originally intended to form the introduction to a work on Theoretical Mechanics. It has seemed best, however, for reasons given in the introductory paragraph, to publish it as a separate volume.

The book is intended as a drill book, and the unusual elaboration of the topics, together with the great number of illustrative examples, should enable the student to overcome most of his difficulties himself. Extreme rigor has not been attempted, but it is hoped that the discussion is free from serious inaccuracies. Many well-known works on the calculus and on mechanics have been drawn upon freely, particularly in the matter of problems, of which a great number is given at the end of each chapter. These have been taken chiefly from the works of Routh, Minchin, Bowser, Price, Todhunter, Lamb, Walton, and Byerly.

The author wishes to acknowledge his indebtedness for helpful criticism and suggestions to Professors P. F. Smith and A. J. Dubois of Yale University, and to Professor E. R. Hedrick of the University of Missouri. His thanks are due also to Professor Smith, and to Professor D. R. Cortis of Northwestern University, formerly Instructor in Mathematics in Yale University, for assistance in proof reading, and to Professor E. E. Lawton of Colby, formerly Assistant in Physics in Yale University, for help in preparing the figures.

O. C. L.

UNIVERSITY OF COLORADO

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THE INTEGRALS OF MECHANICS

1. Introduction. In the study of mechanics certain types of integrals are met with great frequency. These integrals, namely the inertia integrals, those defining mass, and moment and center of mass, are essential in the discussion of the motion and the conditions of equilibrium of systems of particles, and rigid bodies. Their evaluation, however, is purely an application of the integral calculus and has nothing to do with mechanics proper. Their treatment in most works on the calculus is very brief, and their discussion in the midst of a course in mechanics should, in the author's opinion, be avoided. The aim of this little volume is to present an extensive consideration of those integrals suitable for use in courses in the integral calculus and as an introduction to theoretical mechanics. This arrangement should produce a saving of both time and energy: a saving of time by providing numerous applications and problems in the calculus which are useful in mechanics; a saving of energy by preventing one or more breaks in the continuity of the course in mechanics in order to evaluate integrals, and by removing, as far as possible, the liability of certain troublesome confusions which careful teaching does not always prevent. For, when such ideas as *momentum*, *moment of inertia*, *moment of a force*, *moment of mass*, *center of mass or gravity*, and the *force of gravity* are presented for the first time in more or less intimate association, as is usually the case, it is not surprising that beginners often fail to distinguish clearly between those conceptions which are purely mathematical and those which have to do with force and motion.

For some years the author has begun his courses in mechanics with the calculation of centers of mass and moments and ellipsoids

of inertia. The results have been very satisfactory in avoiding the confusion mentioned above, and the review of the calculus thus obtained has largely removed the mathematical difficulties from the later parts of the subject in which the attention should be given to the physical conceptions involved rather than to the manipulation of symbols. The ultimate aim, however, should be to do away entirely with the consideration of these integrals in teaching mechanics, a thing difficult to do unless the student can earlier acquire considerable facility and precision in his use of the calculus.

CHAPTER I

FORMULÆ FROM THE CALCULUS.* THE INTEGRALS OF VOLUME, AREA, LENGTH, AND MASS

2. Volume of any solid. $V = \int dv$. For any solid referred to rectangular coördinates, the element of volume Δv is given by

$$\Delta v = \Delta z \Delta y \Delta x,$$

whence

$$V = \int dv = \iiint dz\,dy\,dx, \tag{1}$$ [Calculus, p. 415]

the limits of integration being determined by the bounding surfaces.

3. Volume of a solid of revolution. Let the X-axis be the axis of revolution, and let the generating area be that included by the axis of X, the ordinates $x = a$ and $x = b$, and the curve $y = f(x)$. Then, from (1),

$$V = \int_a^b \left[\iint dy\,dz\right] dx,$$

or

$$V = \int_a^b \pi y^2 dx = \pi \int_a^b [f(x)]^2 dx. \tag{1}$$ [Calculus, p. 384]

A geometrical representation of this process of integration is obtained as follows:

The integral $\iint dy\,dz$ represents the area of a cross section of the solid perpendicular to the axis of X. This cross section is a circle of radius y and area πy^2. The volume Δv of a thin slice of the solid whose base is πy^2 and whose thickness or height is Δx is approximately $\Delta v = \pi y^2 \Delta x$, and the entire volume is the limit of the sum of the volumes of all these elementary slices or plates, *this process being indicated by the sign of integration.*

* It is assumed that the student is already familiar with the methods expounded in the calculus for the determination of volumes, areas, and lengths. The usual formulæ are given in this chapter merely for convenience of reference. All references to the calculus in this volume are to the work entitled *Elements of the Differential and Integral Calculus*, by W. A. Granville, Ph.D. (Ginn & Company).

Polar coördinates. Suppose the equation of the curve is given in polar coördinates. Then the element of the area revolved is approximately

$$\Delta A = \rho \Delta\rho \Delta\theta, \qquad \text{[Calculus, p. 406]}$$

and its radius is $\rho \sin\theta$. This element of area will therefore generate a ring whose volume is approximately

$$\Delta V = 2\,\pi\rho^2 \sin\theta\, \Delta\rho\Delta\theta.$$

For the ring may be cut through by a plane and its volume evidently lies between that of two prisms each having the base $\rho\Delta\rho\Delta\theta$ and whose altitudes are respectively the inner circumference of the ring, $2\,\pi\rho \sin\theta$, and the outer circumference, $2\,\pi(\rho + \Delta\rho)\sin(\theta + \Delta\theta)$. The volume of the smaller prism is therefore $\rho\Delta\rho\Delta\theta \cdot 2\,\pi\rho\sin\theta$, which is the approximate expression for ΔV given above.

The total volume V of the solid is given as before by the limit of the sum of all such rings, that is, by

$$\text{limit}\, \Sigma\Delta V = 2\,\pi\, \text{limit}\, \Sigma\Sigma\rho^2 \sin\theta\, \Delta\rho\Delta\theta.$$

(2) $$\boldsymbol{V = 2\,\pi \iint \rho^2 \sin\theta\, d\rho d\theta.}$$

4. Solid of known cross section. Consider any solid possessing the property that the area of any cross section A parallel to a fixed plane is a known function of the distance from that plane. Then if the fixed plane is YOZ and the known function $\phi(x)$, the volume is

(1) $$\boldsymbol{V = \int \left[\iint dydz\right] dx = \int \phi(x)\, dx.} \qquad \text{[Calculus, p. 420]}$$

The process of finding the volume of such a solid may be treated in a manner analogous to that employed for a solid of revolution. Any cross section is known and is equal to $A = \iint dydz = \phi(x)$. Regarding this area as the base of a thin cylinder or thin plate of height or thickness Δx, the element of volume is

$$\Delta v = A\Delta x = \phi(x)\,\Delta x,$$

and the limit of the sum of all the volumes of these thin slices constitutes the volume of the entire solid, the limit of the sum being given by the integral sign in equation (**1**).

5. Area of any surface. $\boldsymbol{S = \int dS}$. For any surface referred to rectangular coördinates the element of surface is

$$\Delta S = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, \Delta x \Delta y, \qquad \text{[Calculus, p. 412]}$$

and

(1) $$\boldsymbol{S = \iint \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy,}$$

where the letters x, y, z may be permuted.

6. Surface of revolution. If the axis of revolution is OX and the equation of the generating curve is $y = f(x)$, then, approximately,

$$\Delta S = 2\pi y \Delta s = 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\Delta x,$$

and

(1) $$S = 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx,$$ [Calculus, p. 389]

where x and y may be interchanged if desired.

It is instructive to regard the process of finding the area as follows. Let Δs be a small arc of the generating curve. Then by the revolution of the curve $y = f(x)$ around the axis OX, the arc Δs will generate an element of area ΔS of the surface of revolution, and this element of area will be approximately the convex surface of the frustum of a cone of revolution, the circumference of whose median section is $2\pi y$ and whose slant height is Δs; that is, approximately,

$$\Delta S = 2\pi y \Delta s.$$

The total surface is the limit of the sum of all these elementary surfaces.

If the generating curve is given in polar coördinates, then, approximately,

$$\Delta S = 2\pi y \Delta s = 2\pi \rho \sin\theta \Delta s,$$

whence

(2) $$S = 2\pi \int \rho \sin\theta \sqrt{1 + \rho^2\left(\frac{d\theta}{d\rho}\right)^2}\,d\rho.$$

7. Plane areas. $A = \int dA$. For any area referred to rectangular coördinates $\Delta A = \Delta y \Delta x$, whence

(1) $$A = \iint dy\,dx.$$ [Calculus, p. 403]

If the area is referred to polar coördinates, $\Delta A = \rho\Delta\rho\Delta\theta$ approximately; whence

(2) $$A = \iint \rho\,d\rho\,d\theta.$$ [Calculus, p. 406]

8. Lengths of curves. $s = \int ds$. For any plane curve referred to rectangular coördinates we have, approximately, $\Delta s^2 = \Delta x^2 + \Delta y^2$, whence

(1) $$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$ [Calculus, p. 380]

In polar coördinates, approximately, $\Delta s^2 = \Delta\rho^2 + \rho^2\Delta\theta^2$ [Calculus, p. 382], whence

$$(2) \qquad s = \int\sqrt{1+\rho^2\left(\frac{d\theta}{d\rho}\right)^2}\,d\rho = \int\sqrt{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2}\,d\theta.$$

For any skew curve in space, approximately,

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2,$$

whence

$$(3) \qquad s = \int\sqrt{dx^2+dy^2+dz^2},$$

which is easily evaluated when the equation of the curve is given in parametric form either in rectangular or in polar coördinates.

9. The mass integral, $M = \int dm$. Density.

I. Homogeneous solids. A solid is said to be *homogeneous* or of *uniform density* when equal volumes of it have equal masses. In such a body, therefore, the ratio of the mass to the volume of different portions is constant, and this constant is called the *density;* that is,

$$(1) \qquad Density = \frac{Mass}{Volume}.$$

Let Δm be the mass of the element of volume Δv of any such homogeneous solid referred to rectangular coördinates. Also let τ denote the density and M the total mass. Then, by (1),

$$\Delta m = \tau\Delta v,$$

and

$$M = \underset{\Delta m = 0}{\text{limit}}\ \Sigma\Delta m = \underset{\Delta v = 0}{\text{limit}}\ \Sigma\tau\Delta v = \underset{\Delta x = \Delta y = \Delta z = 0}{\text{limit}}\ \Sigma\tau\Delta z\Delta y\Delta x.$$

Hence, since τ *is constant*,

$$(2) \qquad M = \int dm = \tau\int dv = \tau\iiint dz\,dy\,dx = \tau V.$$

II. Non-homogeneous solids. If the body is not homogeneous, it becomes necessary to introduce the conception of **density at each point of the solid**, which may be done as follows. Let us suppose again that we have rectangular coördinates, and let $P(x, y, z)$ be a vertex of an element of volume $\Delta v = \Delta x\Delta y\Delta z$. Then if Δm is the mass of this element, the quotient $\frac{\Delta m}{\Delta v}$ is called the mean density of the element.

The density at the point $P(x, y, z)$ is defined as the limit of the mean density as the three dimensions of Δv approach zero. That is,

$$\textbf{Density at the point } P(x, y, z) = \tau = \lim_{\Delta x = \Delta y = \Delta z = 0} \frac{\Delta m}{\Delta v},$$

or

(3) $$\tau = \frac{dm}{dv}.$$

Hence

(4) $$M = \int dm = \int \tau dv = \iiint \tau dz dy dx.$$

For non-homogeneous solids τ is a function of (x, y, z) and hence cannot be taken from under the sign of integration.

The necessary formulæ in polar coördinates may be written by the student as an exercise.

For a non-homogeneous solid equation (1) defines the *mean density* which is denoted by $\bar{\tau}$. Hence for such a solid

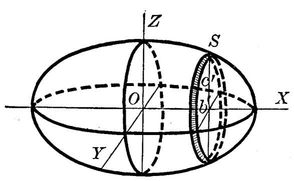
(5) $$\textit{Mean density} = \bar{\tau} = \frac{\textit{Mass}}{\textit{Volume}},$$

or

(6) $$\bar{\tau} = \frac{\int \tau dv}{\int dv} = \frac{\iiint \tau dz dy dx}{\iiint dz dy dx}.$$

Example 1. *To find the mass and mean density of a semi-ellipsoid, the density at each point of which varies as the cube of its distance from one of the principal planes.*

Let the equation of the ellipsoid be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and consider the part cut off by the plane YZ. If the density varies as the cube of the distance from YZ, put $\tau = kx^3$, where k is some constant.



Now the area of any section S parallel to the fixed plane YZ at a distance x is a known function of x. For if we set $x = x$ (that is, a fixed value) in the equation of the ellipsoid above, the equation of S becomes

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \left(1 - \frac{x^2}{a^2}\right),$$

where now x is to be regarded as constant. Hence the semi-axes of the elliptic section S are $b' = b\sqrt{1 - \frac{x^2}{a^2}}$ and $c' = c\sqrt{1 - \frac{x^2}{a^2}}$, and its area is $\pi b'c' = \pi bc\left(1 - \frac{x^2}{a^2}\right)$.

Regarding the semi-ellipsoid as made up of a number of similar thin slices or plates each of area $\pi bc\left(1-\frac{x^2}{a^2}\right)$, of density τ, and thickness Δx, the mass of one of them is approximately

$$\Delta m = \tau \cdot \pi bc\left(1-\frac{x^2}{a^2}\right)\Delta x,$$

and the whole mass is $\quad M = \int dm = \int_0^a kx^3 \cdot \pi bc\left(1-\frac{x^2}{a^2}\right)dx,$

or

$$M = \frac{\pi a^4 bc}{12}k.$$

The mean density is, by (6), p. 7,

$$\bar{\tau} = \frac{\iiint \tau dzdydx}{\iiint dzdydx} = \frac{\frac{\pi}{12}a^4bck}{\frac{2}{3}\pi abc} = \frac{a^3}{8}k.$$

It is seen that the mean density $\bar{\tau}$ and the density τ are the same in the plane $x = \frac{a}{2}$.

Example 2. *To find the mass and mean density of the solid bounded by the surface* $x^2z^2 + a^2y^2 = c^2x^2$ *and the planes* $x = 0$ *and* $x = 2a$ *if the density at each point varies as* xyz.

Here $\tau = kxyz$, and integrating over the first octant only, we have

$$M = 4\int_0^{2a}\int_0^{\frac{cx}{a}}\int_0^{\frac{a}{x}\left(\frac{c^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}} kxyz \cdot dzdydx = 2k\int_0^{2a} xdx\int_0^{\frac{cx}{a}}\frac{a^2}{x^2}\left(\frac{c^2x^2}{a^2}-y^2\right)ydy,$$

or

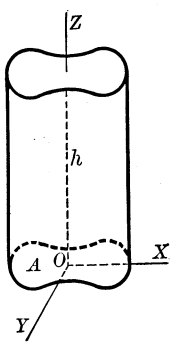
$$M = 2k\int_0^{2a}\frac{c^4x^2}{4a^2}xdx = 2a^2c^4k,$$

and

$$\bar{\tau} = \frac{kac^2}{2\pi}.$$

10. Application to particular solids.

I. Thin flat plate (*lamina*). Suppose the given solid is any right cylinder whose base A lies in the plane XY. Suppose further that the density τ is a function of x and y only; that is, the density is constant along any line parallel to OZ, but may be different for different lines. Let the height of the cylinder be h. Then, by (4), p. 7,



$$M = \iiint \tau dzdydx = \int_0^h dz\left[\iint_A \tau dxdy\right]$$
$$= h\iint_A \tau dxdy,$$

the integration now extending only over the base A. For the volume we have

$$V = h\iint\limits_{A} dydx,$$

and since $\int dA = \iint dydx$, we may write

$$\boldsymbol{M = h\int \tau dA.} \tag{1}$$

$$V = h\int dA = hA. \tag{2}$$

Hence (6), p. 7, becomes

$$\bar{\boldsymbol{\tau}} = \frac{\int \boldsymbol{\tau dA}}{\int \boldsymbol{dA}}. \tag{3}$$

From the manner in which equation (3) was obtained it is evident that it applies to all cylinders, including those whose height is very small, that is, to thin flat plates or laminæ. For such solids (3) is important. Hence

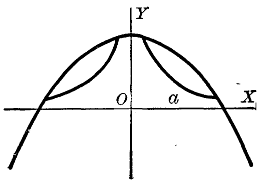
(4) **Mean density of a thin flat plate** $= \bar{\boldsymbol{\tau}} = \dfrac{\int \boldsymbol{\tau dA}}{\int \boldsymbol{dA}}.$

In this formula the density τ and the element of area (1), (2), p. 5, may be expressed either in rectangular or in polar coördinates.

Strictly speaking, density is not a property of a surface or of an area. But since any thin flat plate is completely described by its surface and the variation of the density upon that surface, formula (3) is often said to define the mean density of the surface itself. As was seen in deriving (3), the thickness of the plate, which was taken as constant, divides out. Hence we may, if we choose, regard the two conceptions — *mean density of a thin flat plate or lamina* and *mean density of an area* — as interchangeable.

Example 1. *Given the parabola $x^2 + ay = a^2$, to find the mean density of the area cut off by the X-axis when the density varies as xy.*

In this case $\tau = kxy$, where the absolute values of x and y are to be considered regardless of sign, as density is never negative. Hence, by (**4**),



$$\bar{\tau} = \frac{\int \tau dA}{\int dA} = \frac{\iint kxy \cdot dydx}{\iint dydx},$$

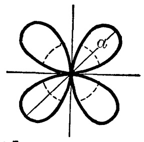
$$\bar{\tau} = \frac{k\int_0^{+a} xdx \int_0^{\frac{a^2-x^2}{a}} ydy}{\int_0^{+a} dx \int_0^{\frac{a^2-x^2}{a}} dy} = \frac{\frac{ka^4}{12}}{\frac{2a^2}{3}} = \frac{ka^2}{8}.$$

Hence the mean density is the same as the actual density at any point the product of whose coördinates equals $\frac{a^2}{8}$; that is, they are the same along the curve $xy = \frac{a^2}{8}$, which is a rectangular hyperbola, a portion of which is shown in the figure. There is a similar hyperbola in the left quadrant.

Example 2. *Given the curve $\rho = a \sin 2\theta$, to find the mean density of one loop when the density varies as ρ.*

Here we have $\tau = k\rho$, and the limits of integration are evidently 0 and $\frac{\pi}{2}$ for θ, and 0 and $a \sin 2\theta$ for ρ. The mean density is then, by (**4**),

$$\bar{\tau} = \frac{\int \tau dA}{\int dA} = \frac{\iint k\rho \cdot \rho d\rho d\theta}{\iint \rho d\rho d\theta}.$$



$$\iint k\rho \cdot \rho d\rho d\theta = k\int_0^{\frac{\pi}{2}} d\theta \int_0^{a \sin 2\theta} \rho^2 d\rho = \frac{ka^3}{3}\int_0^{\frac{\pi}{2}} \sin^3 2\,\theta d\theta$$

$$= \frac{ka^3}{3}\left[-\frac{\sin^2 2\,\theta \cos 2\,\theta}{6} - \frac{1}{3}\cos 2\,\theta\right]_0^{\frac{\pi}{2}} = \frac{2\,a^3k}{9}.$$

$$\iint \rho d\rho d\theta = \int_0^{\frac{\pi}{2}} d\theta \int_0^{a \sin 2\theta} \rho d\rho = \frac{\pi a^2}{8}.$$

$$\therefore\ \bar{\tau} = \frac{16}{9}\,\frac{ak}{\pi}.$$

Here we see that the actual and the mean density coincide on the circle whose radius is $\frac{16\,a}{9\,\pi}$, which is indicated by the dotted line in the figure.

II. Thin wires or material curves. A rigorous discussion of equations (**4**) and (**6**), p. 7, as applied to curves or to thin wires presents difficulties which are beyond the scope of this book. The formulæ applicable to this case may, however, be derived by the following approximate method.

Suppose we have a thin wire which differs very little from a curve c, its cross sections having a small constant area σ. Then if Δs represents a small arc of c, the element of volume of the wire may be taken as $\Delta v = \sigma \Delta s$. Hence, from equation (6), p. 7,

$$\bar{\tau} = \frac{\int \tau \sigma ds}{\int \sigma ds};$$

and since σ is constant,

(5) **Mean density of any material curve** $= \bar{\tau} = \dfrac{\int \tau ds}{\int ds}$.

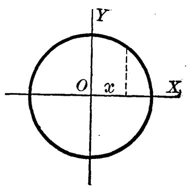
The density τ and the element of arc, ds, may be expressed either in rectangular or in polar coördinates.

Since a thin wire is completely described by its curve and the variation of the density along the curve, the expression *mean density of a curve* is often used, although in reality the expression is meaningless except as defined in equation (5) above.

Example 1. *To find the mean density of a quarter circumference of a circle* $x^2 + y^2 = a^2$ *if the density varies as* x.

Here $\tau = kx$ and the mean density is

$$\bar{\tau} = \frac{\int \tau ds}{\int ds} = \frac{\int kx \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}{\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}.$$



From the equation of the curve, $\left(\dfrac{dy}{dx}\right)^2 = \dfrac{x^2}{y^2}$,

and $$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{y^2}} = \sqrt{\frac{x^2 + y^2}{y^2}} = \frac{a}{y}.$$

$$\therefore \int_0^a ka \frac{x}{y} dx = ka \int_0^a \frac{x dx}{\sqrt{a^2 - x^2}} = ka \left[-\sqrt{a^2 - x^2} \right]_0^a = ka^2.$$

Similarly

$$\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = a \int_0^a \frac{dx}{y} = a \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = a \left[\sin^{-1} \frac{a}{x} \right]_0^a = \frac{a}{2} \pi.$$

$$\therefore \bar{\tau} = \frac{ka^2}{\frac{1}{2} a\pi} = \frac{2\,ak}{\pi}.$$

It is easily seen that the mean density of the quarter circumference will be the same as the actual density at the point whose abscissa is $x = \dfrac{2\,a}{\pi}$.

PROBLEMS

The following list of problems may be used not only in connection with the present chapter but also with Chapters II and III when problems are desired for which the answers are not given.

I. CURVES

1. A straight line of length l has a density which varies as the nth power of the distance from one end. Find the mean density and the point at which the mean and actual densities are equal.

Ans. $\bar{\tau} = \dfrac{kl^n}{n+1}$.

Mean density = Density at a distance $l(n+1)^{-\frac{1}{n}}$ from the end.

2. Find the length of the arc of a sector of the circle $\rho = a\cos\theta$ between $-\theta$ and $+\theta$. *Ans.* $s = 2a\theta$.

3. Find the perimeter of the cardioid $\rho = a(1-\cos\theta)$. Find also the mean density when the density varies as ρ. *Ans.* $s = 8a$.

4. Find the length of the tractrix $ydx = -(a^2 - y^2)^{\frac{1}{2}}dy$. *Ans.* $s = a\log\dfrac{a}{y}$.

5. Find the length of the arc of the helix $x = a\cos\theta$, $y = a\sin\theta$, $z = ka\theta$ from the origin to $\theta = \theta_1$. *Ans.* $s = a\theta_1\sqrt{1+k^2}$.

6. Find the length of the first spire in the spiral of Archimedes $\rho = a\theta$.

Ans. $s = a\pi\sqrt{1+4\pi^2} + \dfrac{a}{2}\log\left(2\pi + \sqrt{1+4\pi^2}\right)$.

7. Find the arc of the catenary $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ between $x = -a$ and $x = a$. Find also the mean density when the density varies as $e^{-\frac{x}{a}}$. *Ans.* $s = a\left(e - \dfrac{1}{e}\right)$.

8. Find the length of the arc of the parabola $y^2 = 4ax$ included between the origin and the point where it is cut by an ordinate through the focus. Suppose the density varies as $y^{\frac{1}{2}}$; find the mean density and the point at which the mean density coincides with the actual density. *Ans.* $s = a\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right]$.

9. Find the length of the hyperbolic spiral $\rho\theta = a$ between $\theta = \frac{5}{12}$ and $\theta = \frac{3}{4}$.

Ans. $s = \frac{14}{15} + \log\frac{4}{3}$.

10. Find the length of the arc of the logarithmic spiral $\rho = e^{a\theta}$ between the points $(1, 0)$ and (ρ_1, θ_1). *Ans.* $s = \dfrac{\rho_1 - 1}{a}\sqrt{1+a^2}$.

11. Find the length of a quadrantal arc of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ans. $s = \dfrac{3a}{2}$.

12. Find the length of the arc cut off the semi-cubical parabola $ay^2 = x^3$ by $x = a$. Find the mean density if $\tau = k\left(1 + \dfrac{9x}{4a}\right)^{\frac{1}{2}}$.

Ans. $s = \dfrac{26a\sqrt{13} - 16a}{27}$. $\bar{\tau} = \dfrac{459}{26\sqrt{13} - 16}k$.

13. Find the length of the arc of the cycloid $x = a\,\text{vers}^{-1}\dfrac{y}{a} - (2ay - y^2)^{\frac{1}{2}}$ between two successive cusps. *Ans.* $s = 8a$.

II. AREAS

1. Find the areas of a rectangle and a triangle by double integration. Find also the area of a parallelogram two of whose sides meet at an angle ω.

2. Find the area bounded by $y = 4x + 5a$, the X-axis, and the lines $x = 0$ and $x = 3a$. Find also the mean density if the density varies as $\frac{5}{2}x^{\frac{1}{2}}$.

Ans. $\text{A.} = 33a^2$. $\bar{\tau} = \frac{61}{33}\sqrt{3a} \cdot k$.

3. Find the area included by $y = x^2$, the X-axis, $x = 1$ and $x = 4$. Find the ordinate y_1 of the point for which the mean and actual densities are equal if τ varies as y.

Ans. $\text{A.} = 21$. $y_1 = \dfrac{4^5 - 1}{210}$.

4. Find the area and the mean density of a loop of the lemniscate $\rho^2 = a^2 \cos 2\theta$ if the density varies as $\cos 2\theta$.

Ans. $\text{A.} = \dfrac{a^2}{2}$. $\bar{\tau} = \dfrac{\pi k}{4}$.

5. Find the area and mean density of a quadrant of the circle $x^2 + y^2 = a^2$ if the density varies as xy.

Ans. $\text{A.} = \dfrac{\pi a^2}{4}$. $\bar{\tau} = \dfrac{ka^2}{2\pi}$.

6. Find the area and mean density of a sector of the circle $\rho = a\cos\theta$ between $-\theta$ and $+\theta$ if the density varies as $\sec\theta$.

Ans. $\text{A.} = \dfrac{a^2}{2}(\theta + \frac{1}{2}\sin 2\theta)$. $\bar{\tau} = \dfrac{2k\sin\theta}{\theta + \frac{1}{2}\sin 2\theta}$.

7. Find the area between the hyperbola $xy = c^2$, $x = a$, $x = b$, and the axis of X.

Ans. $\text{A.} = c^2 \log \dfrac{b}{a}$.

8. Find the area between $ax^n = \pm y$, $x = b$, and $x = c$, where $c > b$.

Ans. $\text{A.} = \dfrac{2a}{n+1}(c^{n+1} - b^{n+1})$.

9. Find the area between the X-axis and the curve $y^2 = 2x^4 + 3a^2y^2$.

10. Find the area between the sine curve $y = a\sin\dfrac{x}{a}$ and the axis of X from $x = 0$ to $x = a\pi$.

Ans. $\text{A.} = 2a^2$.

11. Find the area common to the two parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Ans. $\text{A.} = \frac{16}{3}a^2$.

12. Find the area included by $y^2 = ax$ and $x^2 = by$, and the mean density if τ varies as $\sqrt{x}$.

Ans. $\text{A.} = \dfrac{ab}{3}$. $\bar{\tau} = \frac{9}{14}a^{\frac{1}{6}}b^{\frac{1}{3}}k$.

13. Find the area and mean density of that portion of a thin plate included between the tangent circles $\rho = 2a\cos\theta$ and $\rho = 2b\cos\theta$ and the diameter through the point of contact, $a > b$ and the density varying as $\sin^2\theta$.

Ans. $\text{A.} = \dfrac{\pi}{2}(a^2 - b^2)$. $\bar{\tau} = \dfrac{k}{4}$.

14. Find the area between the axes and the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

Ans. $\text{A.} = \dfrac{a^2}{6}$.

15. Find the area of a complete arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.
Ans. A. $= 3\pi a^2$.

16. Find the area between the tractrix $y dx = -(a^2 - y^2)^{\frac{1}{2}} dy$ and its asymptote.
Ans. A. $= \frac{\pi a^2}{2}$.

17. Find the area between the witch $y = \frac{8a^3}{x^2 + 4a^2}$ and its asymptote $y = 0$.
Ans. A. $= 4\pi a^2$.

18. Find the area between the first and second spires of the spiral of Archimedes $\rho = a\theta$.
Ans. A. $= 8a^2\pi^3$.

19. Find the area of the segment of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{2}$.
Ans. A. $= \frac{1}{3}\pi a^2 - \frac{a^2\sqrt{3}}{4}$.

20. Find the area (a) between the parabola $y^2 = 4ax$, the X-axis, and the line $x = a$; (b) between the curve, the Y-axis, and $y = b$.
Ans. (a) A. $= \frac{4}{3}a^2$; (b) A. $= \frac{b^3}{12a}$.

21. Find the area of the cardioid $\rho = 2a(1 - \cos\theta)$. *Ans.* A. $= 6\pi a^2$.

22. Find the area bounded by the semi-cubical parabola $ay^2 = x^3$ and $x = a$.
Ans. A. $= \frac{4}{5}a^2$.

23. Find the area of a sector of the logarithmic spiral $\rho = e^{a\theta}$ between the radii for which $\theta = 0$ and $\theta = \beta$.
Ans. A. $= \frac{e^{2a\beta} - 1}{4a}$.

24. Find the area of the segment of the parabola $y^2 = 4ax$ cut off by the line $3y - 2x = 4a$.
Ans. A. $= \frac{a^2}{3}$.

25. Find the area of the sector of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ cut out by radii through the points whose abscissæ are $x = a$ and $x = \frac{3}{4}a$. Find also the total area.
Ans. Area sector $= \frac{3ab}{16}\sqrt{7} + \frac{ab}{2}\sin^{-1}\frac{3}{4}$. Total area $= \pi ab$.

26. Find the area of the segment of an ellipse cut off by a chord joining the ends of the major and minor axes.
Ans. A. $= \frac{ab}{4}(\pi - 2)$.

27. Find the area of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
[Let $x = a\cos^3\theta$ and $y = a\sin^3\theta$.]
Ans. A. $= \frac{3}{8}\pi a^2$.

28. Find the areas of the three parts into which the curve $x^2 + y^2 = a^2$ is divided by the curve $x^2 - y^2 = a^2$.
Ans. Area middle part $= 2a\left[3\sin^{-1}\sqrt{\frac{2}{3}} + \log(1 + \sqrt{2})\right]$.

III. VOLUMES. SURFACES

1. Find the volume of a parallelopiped by triple integration.

2. Find the volume, mass, and mean density of the sphere $x^2 + y^2 + z^2 = a^2$ where the density varies as the distance from the plane $x = 0$.

Ans. $V. = \frac{4}{3}\pi a^3$. $M. = \frac{\pi k a^4}{2}$. $\bar{\tau} = \frac{3\,ka^4}{8}$.

3. Find the volume bounded by the surfaces $y^2 + x^2 = 4\,z$, $y^2 + x^2 = 3\,x$, and $z = 0$. *Ans.* $V. = \frac{243}{128}\,\pi$.

4. Find the volume and surface of a hemisphere regarding it as a surface of revolution. *Ans.* $V. = \frac{2}{3}\,\pi a^3$. $S. = 2\,\pi a^2$.

5. Find the volume of an octant of an ellipsoid, using the method of Ex. 1, p. 7. *Ans.* $V. = \frac{\pi abc}{6}$.

6. Find the volume and surface of the cone formed by the revolution of the line $hy = ax$ around the X-axis, between $x = 0$ and $x = h$.

Ans. $V. = \frac{h}{3}\pi a^2$. $S. = \pi a\sqrt{h^2 + a^2}$.

7. In Ex. 6, if the density varies as the nth power of the distance from the vertex, find M and $\bar{\tau}$.

8. Required the volume formed by the revolution of the curve $y = a\sin\frac{x}{a}$ about its axis, between $x = 0$ and $x = \frac{1}{2}\,a\pi$. Find also the mass and mean density if τ varies as $\cos\frac{x}{a}$, and the point at which the mean and actual densities are the same.

Ans. $\bar{\tau} = \frac{4\,k}{3\,\pi}$. $V. = \frac{\pi^2 a^3}{4}$. $M. = \frac{\pi a^3 k}{3}$.

9. Required the surface and volume formed by the revolution of a quadrant of a circle about the tangent at its extremity.

Ans. $S. = \pi a^2(\pi - 2)$. $V. = \frac{\pi a^3}{6}(10 - 3\pi)$.

10. Required the surface and volume obtained by the revolution of the tractrix $y^2dx = -(a^2 - y^2)^{\frac{1}{2}}\,ydy$ about the X-axis. *Ans.* $V. = \frac{2}{3}\,\pi a^3$. $S. = 2\,\pi a^2$.

11. Find the volume included by $4x^2 + y^2 + 4z^4 = 4$. *Ans.* $V. = \frac{16\pi}{5}$.

12. Find the volume and surface of a paraboloid of revolution whose generating curve is $y^2 = 4\,ax$, between the vertex and the plane $x = b$.

Ans. $V. = 2\,\pi ab^2$. $S. = \frac{8\,\pi\sqrt{a}}{3}\left[(a + b)^{\frac{3}{2}} - a^{\frac{3}{2}}\right]$.

13. If the density of both the volume and surface in Ex. 12 varies as $(a + x)$, find the mass and the mean density of both the solid and the surface.

14. Find the volume of the cylinder $x^2 + z^2 = a^2$ cut by the plane $y = 0$ and a plane through the Z-axis making an angle ϕ with the XZ-plane.

15. Find the volume and the surface of the frustum of the cone of Ex. 6 made by the plane $x = b$.
Ans. $\text{V.} = \dfrac{\pi a^2 (h - b)}{3 h^2}(b^2 - bh + h^2)$.

16. A cone of the same base and height is cut from a paraboloid of revolution. Find the volume of the remaining solid.
Ans. $\text{V.} = \dfrac{\pi h}{3}(6\, ah - r^2)$.

17. Consider the arc of the parabola $y^2 = 4\, ax$ between the origin and the point $(a, 2\, a)$. Find the volume and surface formed by the revolution of the arc about the line $x = a$.
Ans. $\text{V.} = \frac{16}{15}\pi a^3$.

18. Find the volume and surface formed by revolving the arc of Ex. 17 about the Y-axis.
Ans. $\text{V.} = \frac{2}{5}\pi a^3$.

19. Find the volume formed by revolving the area between the arc of Ex. 17, the X-axis, and the line $x = a$, about the Y-axis.

20. The arc of the catenary $y = \dfrac{a^2}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ between $x = 0$ and $x = a$ is revolved about the Y-axis. Find the surface and volume generated.
Ans. $\text{S.} = 2\,\pi a^2\left(1 - \dfrac{1}{e}\right)$. $\text{V.} = \dfrac{\pi a^3}{2}\left(e + \dfrac{5}{e} - 4\right)$.

21. Find the surface and volume generated by revolving the quadrant of an ellipse about the X-axis.
Ans. $\text{V.} = \frac{2}{3}\pi ab^2$. $\text{S.} = \pi ab\left[\sqrt{1 - e^2} + \dfrac{\sin^{-1} e}{e}\right]$.

22. Find the volume generated by the revolution of a sector of a circle of radius a about one of its extreme radii, the angle between the radii being β.
Ans. $\text{V.} = \frac{2}{3}\pi a^3(1 - \cos\beta)$.

23. Find the volume generated by the revolution of the cissoid $y^2(2\, a - x) = x^3$ about its asymptote.
Ans. $\text{V.} = 2\,\pi^2 a^2$.

24. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ cut out by the cylinder $x^2 + y^2 = ax$.
[Transform x and y to polar coördinates.]
Ans. $\text{V.} = \frac{4}{3}\pi a^3 - \frac{16}{9}a^3$.

25. The cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ revolves about its base. Find volume and surface generated.
Ans. $\text{V.} = 5\,\pi^2 a^3$. $\text{S.} = \frac{64}{3}\pi a^2$.

26. Find the volume and surface if the cycloid revolves about its axis.
Ans. $\text{V.} = \pi a^3\left(\dfrac{3\,\pi^2}{2} - \dfrac{8}{3}\right)$. $\text{S.} = \dfrac{8\,\pi a^2}{3}(3\,\pi - 4)$.

27. Find the volume and surface generated when the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$ revolves about the tangent at its vertex.
Ans. $\text{V.} = \pi^2 a^3$. $\text{S.} = \frac{32}{3}\pi a^2$.

28. Find the volume and surface formed by the revolution of the cardioid $\rho = 2\, a(1 - \cos\theta)$ about its axis.
Ans. $\text{V.} = \frac{64}{3}\pi a^3$. $\text{S.} = \frac{128}{5}\pi a^2$.

29. A right circular cone of height h, radius a, and having an apex angle $2\,\omega$, is cut by two planes through its axis inclined at an angle θ. Find the portion of the surface cut out.
Ans. $\text{S.} = \dfrac{a^2\theta}{2\sin\omega}$.

30. Two cylinders each of radius a intersect with their axes perpendicular. Find their common volume and the surface of one cut out by the other.

Ans. V. $= \frac{16}{3} a^3$. S. $= 8 a^2$.

31. Find the volume and surface generated by the revolution of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about either axis. *Ans.* V. $= \frac{32}{105} \pi a^3$. S. $= \frac{12}{5} \pi a^2$.

32. Find the volume of a solid of revolution where the Z-axis is an axis of symmetry and where the area of any cross section perpendicular to the axis varies, (a) as $(a - z)$; (b) as $(a - z)^3$; (c) as $\sqrt{a - z^2}$; (d) as $\sin (a - z)$.

33. Find the volume included by the coördinate planes and the hyperbolic paraboloid $y = b \frac{(a - x)}{a} \frac{(c - z)}{c}$. *Ans.* V. $= \frac{abc}{4}$.

CHAPTER II

MOMENT AND CENTER OF MASS, AREA AND ARC

11. Moment of mass. We shall have occasion throughout this volume to make frequent use of the term **material particle**, or **particle** simply. By a material particle is meant a portion of matter so small that it may be regarded, without sensible error, as reduced to a point. In other words, it is a weighted point, or a point-mass.*

The moment of mass of a material particle with respect to a given plane is the product of the mass of the particle and its perpendicular distance from the plane.

Thus if m is the mass of a particle at a point P and r is the distance from a fixed plane E to P, then the moment of mass of the particle with respect to E is rm.

Again, suppose there is a system of n material particles of masses

$$m_1, \quad m_2, \quad \cdots, \quad m_n,$$

whose distances from E are respectively

$$r_1, \quad r_2, \quad \cdots, \quad r_n.$$

Then the sum of the moments of mass of the several particles, that is, the value of

$$\Sigma rm = r_1 m_1 + r_2 m_2 + \cdots + r_n m_n, \tag{1}$$

is called the *moment of mass*† *of the system with respect to E.*

The idea of moment of mass may be extended to a continuous solid as follows. Suppose the given solid S to be divided into elements of mass Δm, and let a point P be chosen in each element according to some law; for example, the centers or corresponding vertices of the elements. Let E be a given fixed plane, and let r be the perpendicular distance from E to the point P chosen in Δm.

* *Mathematically*, a material particle is to be regarded as an element in space having four coördinates, x, y, z, and m, the term *mass* being given to the last.

† Since only the first power of the distance enters into the expression for the moment of mass, the term *moment of the first order* is sometimes used as an equivalent.

Then if we multiply each element of mass by the distance from the plane E to the point P within it, and take the sum of all such products, that is, find $\Sigma r\Delta m$ for the solid, the limit of this sum, or

(2) $$\lim_{\Delta m = 0} \Sigma r\Delta m = \int r dm,$$

is called the *moment of mass of the solid S with respect to the plane E.* The moment of mass will be denoted by C_E. Hence we have with respect to any plane E,

(3) $$\textbf{Moment of mass} = C_E^* = \int r dm.$$

It is important to note that an algebraic sign is necessarily attached to r as in analytic geometry of space (Analytic Geometry,† p. 356). That is, it is necessary to assume r to be positive for all points on one side of the plane E and negative for all points on the other side.

12. Plane of symmetry. Since the distance r carries with it an algebraic sign, the conclusion may be drawn at once that *moment of mass with respect to any plane of symmetry is zero.* For at points similarly situated on each side of E, r has the same numerical value, but has opposite signs, and since the mass Δm of each element is the same, the terms of $\Sigma r\Delta m$ cancel in pairs and $C_E = 0$.

13. Standard form of moment of mass. Since each term of $\Sigma r\Delta m$ is the product of a mass by a length, the sum itself must necessarily be the product of a mass, a length, and a numerical factor. This is true not only for a system of particles but, by (2), for a solid also. Hence the moment of mass may always be expressed in the standard form,

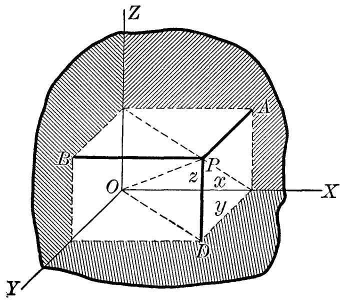
$$C_E = nML,$$

where $n =$ a numerical factor, $M =$ total mass, and $L =$ a length. Dropping the factor n, the right-hand side of the equation becomes the ordinary dimensional formula, which is usually designated by $[ML]$.

* There is no accepted designation for this integral. On account of its frequent occurrence however, some simple notation is very convenient. The letter C is suggested naturally on account of the importance of this integral in the expression for *center of mass,* often called also *center of gravity* or *centroid.*

† References to analytic geometry throughout this volume are to *Elements of Analytic Geometry* by P. F. Smith and A. S. Gale (Ginn & Company).

14. Solids. Rectangular coördinates. In rectangular coördinates $\Delta m = \tau \Delta z \Delta y \Delta x$ (p. 6), and if we choose in turn each coördinate plane as the fixed plane E, then



when E is XY, $r = z$,

when E is XZ, $r = y$,

when E is YZ, $r = x$.

Hence the fundamental **formulæ for moment of mass in rectangular coördinates:**

$$\text{I} \quad \begin{cases} \boldsymbol{C_{xy} = \int z dm = \iiint \tau z dz dy dx.} \\ \boldsymbol{C_{yz} = \int x dm = \iiint \tau x dz dy dx.} \\ \boldsymbol{C_{xz} = \int y dm = \iiint \tau y dz dy dx.} \end{cases}$$

By the aid of equations **I** it is easy to find the moment of mass of a solid with respect to any plane E whose equation in the normal form (Analytic Geometry, p. 348) is

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0.$$

For let r be the distance from the plane E to the point $P(x, y, z)$. Then (Analytic Geometry, p. 357)

$$(1) \qquad r = x \cos \alpha + y \cos \beta + z \cos \gamma - p. \quad \text{Hence}$$

$$\int r dm = \int x \cos \alpha dm + \int y \cos \beta dm + \int z \cos \gamma dm - \int p dm$$
$$= \cos \alpha \int x dm + \cos \beta \int y dm + \cos \gamma \int z dm - p \int dm.$$

$$(2) \qquad \therefore \boldsymbol{C_E = \cos \alpha C_{yz} + \cos \beta C_{xz} + \cos \gamma C_{xy} - pM.}$$

Hence, if we know the moments of mass with respect to three mutually perpendicular, intersecting planes, chosen as the coördinate planes, the moment of mass with respect to any other plane may be found by equation (2).

Theorem. *The moment of mass of any solid with respect to a plane E whose equation is given in rectangular coördinates, is found as follows.*

*First step. Reduce the equation of E to the normal form.**

Second step. Replace each coördinate by the moment of mass with respect to the corresponding coördinate plane, and replace p by p times the total mass of the solid. The value of the right-hand side of the equation gives the desired moment of mass.

Illustrative example. *Required the moment of mass of a rectangular parallelopiped of density 1 with respect to the plane E whose equation is $2x - y + 2z + 3 = 0$; the parallelopiped being bounded by the coördinate planes and by $x = a$, $y = b$, $z = c$.*

The moments of mass with respect to the YZ-plane are given by **I**, p. 20,

$$C_{yz} = \int_0^a \int_0^b \int_0^c x\,dz\,dy\,dx = \frac{a^2bc}{2} = \frac{Ma}{2}.$$

[Since the total mass $M = abc$.]

Similarly $\qquad C_{xz} = \frac{Mb}{2}$ and $C_{xy} = \frac{Mc}{2}$.

The equation of E above reduced to the normal form is

$$-\tfrac{2}{3}x + \tfrac{1}{3}y - \tfrac{2}{3}z - 1 = 0.$$

Hence, by **(2)**, p. 20, the moment of mass with respect to E is

$$C_E = -\frac{2}{3}\frac{Ma}{2} + \frac{1}{3}\frac{Mb}{2} - \frac{2}{3}\frac{Mc}{2} - M = \frac{M}{6}(b - 2a - 2c - 6).$$

15. Moment of mass with respect to an axis or a point. In a manner entirely analogous to the definition of C_E above, we may define the moment of mass of a solid with respect to a line or a point. Thus the moment of mass with respect to the X-axis is

$$C_x = \int r\,dm = \int \sqrt{y^2 + z^2}\,dm,$$

where r in the general formula becomes the distance from $P(x, y, z)$ to the X-axis.

Similarly the moment of mass with respect to the origin is

$$C_0 = \int \sqrt{x^2 + y^2 + z^2}\,dm.$$

However, in the case of solids, moments of mass with respect to lines and points are relatively unimportant; the important case being moments with respect to planes, usually the coördinate planes.

* The equation of a plane is reduced to the normal form as follows (Analytic Geometry, p. 350). Divide the equation of the plane by $\pm\sqrt{A^2 + B^2 + C^2}$. The direction cosines of the normal to the plane are respectively A, B, and C divided by $\pm\sqrt{A^2 + B^2 + C^2}$. The sign of the radical is opposite to that of D, — the same as that of C if $D = 0$, the same as that of B if $C = D = 0$, or the same as that of A if $B = C = D = 0$.

16. Center of mass. Consider the equation $C_{yz} = \int x dm$, which defines the moment of mass with respect to the YZ-plane (**I**, p. 20). Evidently there exists a distance $\bar{x}$ such that $C_{yz} = \bar{x}M$, where M is the total mass of the solid. Similarly we may write $C_{xz} = \bar{y}M$ and $C_{xy} = \bar{z}M$. Hence there exists a certain point $\bar{P}(\bar{x}, \bar{y}, \bar{z})$, whose coördinates are defined by

(1) $$\bar{x} = \frac{C_{yz}}{M}, \qquad \bar{y} = \frac{C_{xz}}{M}, \qquad \bar{z} = \frac{C_{xy}}{M}.$$

(2) Let $\bar{r} = \bar{x} \cos \alpha + \bar{y} \cos \beta + \bar{z} \cos \gamma - p$ be the distance from the plane E whose equation is $x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0$ to the point $\bar{P}$. The moment of mass of a single particle of mass M at the distance $\bar{r}$, with respect to the plane E, is (p. 18)

(3) $$M\bar{r} = M\bar{x} \cos \alpha + M\bar{y} \cos \beta + M\bar{z} \cos \gamma - pM.$$

Also, by equation (2), p. 20, the moment of mass of any solid with respect to the plane E is

(4) $$C_E = C_{yz} \cos \alpha + C_{xz} \cos \beta + C_{xy} \cos \gamma - pM.$$

If we replace $\bar{x}, \bar{y}, \bar{z}$ in (3) by their values in (1), the right-hand members in (3) and (4) become identical and we have

(5) $$C_E = M\bar{r}.$$

That is, a *single particle* of mass M at the point $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ has the *same* moment of mass with respect to the plane E as the original solid. The point $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ is called the center of mass of the solid.

Theorem. *In every system of particles or in every continuous solid there exists a point $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ at which, if the whole mass of the system of particles or of the continuous solid is concentrated into a single particle, the moment of mass of this particle with respect to any plane is the same as that of the original system or solid.*

The position of the center of mass $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ is independent of the choice of axes. It may be regarded physically as the average position of the matter which is contained in the given solid or system of particles.

Formulæ for center of mass in rectangular coördinates:

$$\text{II} \quad \bar{x} = \frac{C_{yz}}{M} = \frac{\iiint \tau x\,dz\,dy\,dx}{\iiint \tau\,dz\,dy\,dx}; \qquad \bar{y} = \frac{C_{xz}}{M} = \frac{\iiint \tau y\,dz\,dy\,dx}{\iiint \tau\,dz\,dy\,dx};$$

$$\bar{z} = \frac{C_{xy}}{M} = \frac{\iiint \tau z\,dz\,dy\,dx}{\iiint \tau\,dz\,dy\,dx}.$$

If the solid is homogeneous, τ is constant and divides out.

From equation (5) we obtain

$$(6) \qquad \bar{r} = \frac{C_E}{M},$$

which gives the distance $\bar{r}$ from any plane E to the center of mass.

17. Principle of symmetry. Centroidal planes and lines. Since the moment of mass with respect to a plane of symmetry (p. 19) is always zero, it is clear that the center of mass must lie in such a plane. For in equation (6),

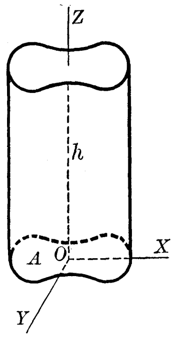
$$\bar{r} = \frac{C_E}{M},$$

$\bar{r}$ is the distance from the plane E to the center of mass, and if E is a plane of symmetry, $C_E = 0$ and hence $\bar{r} = 0$; that is, the center of mass lies in the plane.

It is also evident that if a homogeneous solid is symmetrical with respect to two planes, it is symmetrical with respect to the line formed by their intersection. Such a line is an **axis of symmetry** and passes through the center of mass. The intersection of two axes of symmetry is a **center of symmetry** and this point is the center of mass. Thus the center of mass of a homogeneous right circular cylinder lies on its axis halfway between its bases. The center of mass of such solids as a homogeneous cube, ellipsoid, sphere, etc., is at the geometrical center* or center of volume.

* By geometrical center is meant a center of symmetry, that is, a point bisecting all chords of the solid passing through it.

The center of mass is often called the **centroid** of the body, and planes and lines passing through the centroid are called respectively **centroidal planes** and **centroidal lines.*** Every plane or line of symmetry passes through the center of mass or centroid, but not every centroidal plane or centroidal line is a plane or line of symmetry.



18. Illustrative examples.

1. *Homogeneous right cylinder.* Given any homogeneous right cylinder of height h whose base is in the plane XY. Let A represent the area of the base and V the volume. Then we have, **I**, p. 20,

$$C_{xy} = \tau \iiint z dz dy dx = \tau \int_0^h z dz \Big[\iint_A dx dy\Big] = \tau \frac{h^2}{2} A,$$

$\Big[$Since $\iint dxdy$ is the area of a section of the cylinder for $z = \text{constant}.\Big]$

or

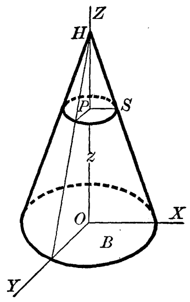
III $$C_{xy} = \tau \frac{h}{2} V = M \frac{h}{2}.$$

Theorem. *The moment of mass of any homogeneous right cylinder with respect to the plane of its base is equal to the product of the total mass M by half the altitude.*

If the point O is the center of symmetry of the base A, then OZ is an axis of symmetry and $C_{yz} = C_{xz} = 0$.

Center of mass. If OZ is an axis of symmetry, $\bar{x}$ and $\bar{y}$ of **II**, p. 23, are evidently zero. But

IV $$\bar{z} = \frac{C_{xy}}{M} = \frac{M\frac{h}{2}}{M} = \frac{h}{2}.$$



Theorem. *The center of mass of any homogeneous right cylinder having an axis of symmetry lies on the axis† at a point halfway between its bases.*

2. *Homogeneous right cone or pyramid.* Let the figure represent a homogeneous cone whose base is in the XY-plane and whose altitude is $h = OH$. Also let S be a section parallel to the base B at the height $OP = z$. Then if OZ is an axis of symmetry, $C_{xz} = C_{yz} = 0$ and

$$C_{xy} = \tau \iiint z dz dy dx = \tau \int_0^h z dz \Big[\iint_S dx dy\Big].$$

* Throughout this work the term *center of mass* is used in preference to either of the terms *center of gravity* or *centroid*. The adjective *centroidal*, however, applied to planes and lines which pass through the center of mass, is very convenient and does away with an awkward circumlocution, all the more objectionable because of its necessarily frequent repetition.

† The axis of any homogeneous cylinder is the line drawn through the center of symmetry (p. 23) of the base parallel to the elements of the cylinder.

But $\iint_S dxdy =$ area of section S at height $OP = z$,

and $$\text{area } B : \text{area } S = (OH)^2 : (PH)^2 = h^2 : (h-z)^2.$$

Hence $$S = \frac{B(h-z)^2}{h^2},$$

and

V $$\boldsymbol{C_{xy} = \frac{B\tau}{h^2}\int_0^h (h-z)^2 z dz = \frac{B\tau h^2}{12} = M\frac{h}{4}}.$$

Precisely the same reasoning applies also to a pyramid.

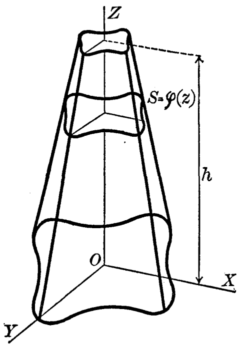
Theorem. *The moment of mass of any homogeneous right cone or pyramid relative to the plane of its base is equal to the entire mass times one fourth its height.*

Center of mass. The center of mass is given at once by the formulæ **II**, p. 23. Evidently $C_{xz} = C_{yz} = 0$, and hence $\bar{x} = \bar{y} = 0$. But

VI $$\boldsymbol{\bar{z} = \frac{C_{xy}}{M} = \frac{M\frac{h}{4}}{M} = \frac{h}{4}}.$$

Theorem. *The center of mass of any homogeneous right cone or pyramid having an axis of symmetry is situated on the axis* at one fourth the distance from the base to the vertex.*

3. *Solid of known cross section.* Let figure represent a solid whose cross sections parallel to a fixed plane are known as a function of the distance from that plane. This general class of solids includes prisms, pyramids, solids of revolution, and many others. Take the plane XY as the fixed plane. Then the area of a cross section parallel to XY at a distance z is, by hypothesis, a known function of z, $\phi(z)$. Hence, if the solid is homogeneous, the moment of mass with respect to the plane XY is (**I**, p. 20)



$$C_{xy} = \iiint \tau z dxdydz = \tau\int_0^h zdz\left[\iint_S dxdy\right].$$

But $\iint dxdy =$ area of any section S for which z is constant. Hence $\iint dxdy = \phi(z)$ and

(1) $$C_{xy} = \tau\int_0^h z\cdot\phi(z)\,dz.$$

(2) Since $$M = \tau\int_0^h \phi(z)\,dz,$$ (cf. Ex. 1, p. 7)

the center of mass is given by

(3) $$\bar{z} = \frac{C_{xy}}{M} = \frac{\int_0^h z\phi(z)\,dz}{\int_0^h \phi(z)\,dz}.$$

Assuming, as before, that z is an axis of symmetry, $\bar{x} = \bar{y} = 0$. A single integration is sufficient to determine either C_{xy} or M.

* The axis of any homogeneous cone or pyramid is the line drawn from the center of symmetry of the base to the vertex.

Since the directions of the coördinate axes are arbitrary, it is evident that z in **(3)** may be replaced by either x or y. Thus if the fixed plane is designated by YZ, the area of any cross section parallel to YZ is $\phi(x)$. That is, we may have

$$\textbf{(4)} \qquad \bar{x} = \frac{C_{yz}}{M} = \frac{\int_0^a \phi(x)\,x\,dx}{\int_0^a \phi(x)\,dx} \quad \text{or} \quad \bar{y} = \frac{C_{xz}}{M} = \frac{\int_0^b \phi(y)\,y\,dy}{\int_0^b \phi(y)\,dy}.$$

When the solid is not homogeneous the formulæ are slightly modified. Thus in **(1)** and **(2)** if τ is also a function of the distance from the fixed plane XY,

$$\textbf{(5)} \qquad C_{xy} = \int_0^h \tau\phi(z)\,z\,dz$$

and

$$\textbf{(6)} \qquad M = \int_0^h \tau\phi(z)\,dz.$$

$$\textbf{(7)} \qquad \therefore\ \bar{z} = \frac{C_{xy}}{M} = \frac{\int_0^h \tau\phi(z)\,z\,dz}{\int_0^h \tau\phi(z)\,dz},$$

where z may be replaced by x or y as before.

Again, if τ is a function of x or y, or x and y, (1) becomes

$$\textbf{(8)} \qquad C_{xy} = \int_0^h z\,dz\left[\iint\limits_S \tau\,dx\,dy\right] = \int_0^h \psi(z)\,z\,dz,$$

where now $\psi(z) = \iint\limits_S \tau\,dx\,dy$ is known and the formulæ corresponding to **(6)** and **(7)** are obvious.

Theorem. *If the areas of parallel cross sections and the density at each point of a solid are known functions of the distance of these cross sections from a given plane E, then the distance from E to the center of mass is given by*

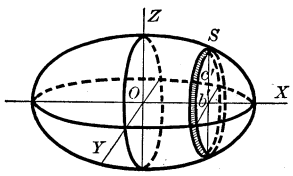
$$\textbf{VII} \quad \bar{x} = \frac{\int \tau x\phi(x)\,dx}{\int \tau\phi(x)\,dx}, \text{ or } \bar{y} = \frac{\int \tau y\phi(y)\,dy}{\int \tau\phi(y)\,dy}, \text{ or } \bar{z} = \frac{\int \tau z\phi(z)\,dz}{\int \tau\phi(z)\,dz},$$

according as the given plane E is YZ, ZX, or XY. The important point is that in **VII** only two integrations are necessary.

The following examples 4–6 are particular cases of **VII**.

4. *Homogeneous ellipsoid.* Take the equation of the ellipsoid as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



and consider the part to the right of the plane YZ. It is evident from the symmetry of the figure that $C_{xy} = C_{xz} = 0$. C_{yz} is obtained as follows.

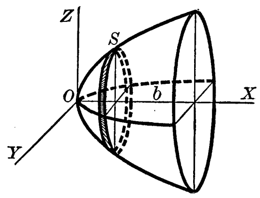
The area of any cross section S parallel to YZ is (p. 7) $\pi b'c' = \pi bc\left(1 - \frac{x^2}{a^2}\right)$.

Hence in **VII**, p. 26, we have $\phi(x) = \pi bc\left(1-\frac{x^2}{a^2}\right)$.

$$\therefore\ \bar{x} = \frac{\tau\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)xdx}{\tau\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx} = \frac{\frac{1}{4}\tau\pi a^2bc}{\frac{2}{3}\tau\pi abc} = \frac{3}{8}a.$$

It is evident that $\bar{y} = \bar{z} = 0$ from considerations of symmetry.

5. *Solids of revolution.* A solid of revolution is a less general case of **VII**, but one, however, of great importance. In the case of such solids the problem of finding the moment and center of mass is considerably simplified. We need not discuss the general case, as the following typical example will make plain the process to be followed in all.



Given the paraboloid formed by the revolution of the curve $y^2 = 4\,ax$ about the X-axis, to find the moment and center of mass of the portion cut off by the plane $x = b$.

Suppose the density varies as the distance from the plane $x = b$.

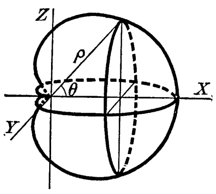
Hence $\tau = k(b - x)$. The cross section S being in this case a circle of radius y, we have $\phi(x) = \pi y^2 = \pi \cdot 4\,ax$. Hence, by **VII**,

$$\bar{x} = \frac{\int_0^b k(b-x)(\pi\cdot 4\,ax)\,xdx}{\int_0^b k(b-x)(\pi\cdot 4\,ax)\,dx} = \frac{b}{2}.$$

Also $\bar{y} = \bar{z} = 0$ by symmetry. The mean density (p. 7) is

$$\bar{\tau} = \frac{\int_0^b k(b-x)(\pi\cdot 4\,ax)\,dx}{\int_0^b \pi\cdot 4\,axdx} = \frac{b}{3}k.$$

6. *To find the moment and center of mass of the homogeneous solid formed by the revolution of the cardioid $\rho = 2\,a(1+\cos\theta)$ about the X-axis.* This example illustrates the use of polar coördinates. Since OX is an axis of symmetry, in **VII**, $\phi(x) = \pi z^2 = \pi\cdot\rho^2\sin^2\theta$. Also $x = \rho\cos\theta$ and $dx = d\rho\cos\theta - \rho\sin\theta d\theta$. Since θ varies from 0 to π, we have, by **VII**,



$$\bar{x} = \frac{\int_0^\pi \pi\cdot\rho^2\sin^2\theta\cdot\rho\cos\theta\,(d\rho\cos\theta - \rho\sin\theta d\theta)}{\int_0^\pi \pi\rho^2\sin^2\theta\,(d\rho\cos\theta - \rho\sin\theta d\theta)}.$$

From the equation of the curve, $d\rho = -2\,a\sin\theta d\theta$. Substitute for ρ and $d\rho$ their values in terms of θ and simplify.

$$\bar{x} = \frac{2\,a\int_0^\pi(1+\cos\theta)^3(1-\cos^2\theta)(2\sin\theta\cos^2\theta d\theta + \cos\theta\sin\theta d\theta)}{\int_0^\pi(1+\cos\theta)^2(1-\cos^2\theta)(2\sin\theta\cos^2\theta d\theta + \cos\theta\sin\theta d\theta)} = -\frac{8\,a}{5}.$$

When the parentheses are expanded and multiplied together the integration termwise becomes very simple.

As an example of a case where there is no plane of symmetry consider the following problem.

7. *To find the moments and center of mass of the homogeneous solid bounded by the surface $z^2 = xy$ and the planes $x = a$, $y = b$, and $z = 0$.*

$$C_{xy} = \int z dm = \tau \iiint dz dy dx = \tau \int_0^a \int_0^b \int^{\sqrt{xy}} dz dy dx = \frac{a^2 b^2}{8} \tau.$$

Similarly we find

$$C_{yz} = \tau \tfrac{4}{15} a^{\frac{5}{2}} b^{\frac{3}{2}}; \quad C_{xz} = \tau \tfrac{4}{15} b^{\frac{5}{2}} a^{\frac{3}{2}}; \quad M = \tfrac{4}{9} \tau a^{\frac{3}{2}} b^{\frac{3}{2}},$$

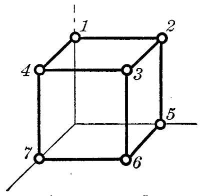
and hence the center of mass is

$$\bar{x} = \frac{C_{yz}}{M} = \frac{3}{5} a; \quad \bar{y} = \frac{C_{xz}}{M} = \frac{3}{5} b; \quad \bar{z} = \frac{C_{xy}}{M} = \frac{9}{32} \sqrt{ab}.$$

PROBLEMS

1. At each corner but one of a unit cubical frame without weight are situated material particles of equal mass. Find the moment of mass with respect to any side of the cube; also find the center of mass, assuming the origin at the unweighted corner.

Ans. $\bar{x} = \bar{y} = \bar{z} = \frac{4}{7}$. Mom. of mass $= 3m$ or $4m$.



2. Find the center and moments of mass if the particles have masses proportional to the numbers beside them.

3. Find the center of mass and moments of mass of a hemisphere whose density varies as x^2, assuming the base in the YZ-plane and the origin at the center of the base. *Ans.* $\bar{x} = \frac{5}{8} a$.

4. Find the center of mass of the cone formed by revolving the line $hy = ax$ around the X-axis between $x = 0$ and $x = h$. *Ans.* $\bar{x} = \frac{3}{4} h$.

5. Suppose the density in (4) varies as the nth power of the distance from the vertex; find the center of mass. *Ans.* $\bar{x} = \dfrac{n+3}{n+4} h$.

6. Find the center of mass of an octant of an ellipsoid.

Ans. $\bar{x} = \frac{3}{8} a$; $\bar{y} = \frac{3}{8} b$; $\bar{z} = \frac{3}{8} c$.

7. Find the moments of mass and the center of mass for the solid included by the surfaces $y^2 + x^2 = 4z$, $y^2 + x^2 = 3x$, and $z = 0$.

8. The area bounded by the lines $y = 0$, $x = a$, and the curve $y^2 = 4ax$ is revolved about the Y-axis. Find the volume and the center of mass of the solid formed. *Ans.* $\bar{y} = \frac{5}{6} a$.

9. Find the moment of mass of an octant of an ellipsoid of density 1 with respect to the plane whose equation is $4x - 3y + z - 2 = 0$.

Ans. $C_E = \dfrac{\pi abc}{12\sqrt{26}} (12a - 9b + 3c - 4)$.

10. Find the moments of mass of the hemisphere in Ex. 3 and of the cone in Ex. 4 with respect to the plane whose equation is $x - y + 2z - 3 = 0$.

19. Moment of area. The definition of the moment of area of any surface, plane or curved, is precisely analogous to that already given for moment of mass (p. 18). The given surface is divided into elements of area ΔS, and a point is selected in each element. Then supposing ΔS to approach zero, the limit of the sum of the products formed by multiplying each ΔS by the distance r of the corresponding point from a fixed *point*, *axis*, or *plane E*, is called the moment of area with respect to the point, line, or plane as the case may be. That is,

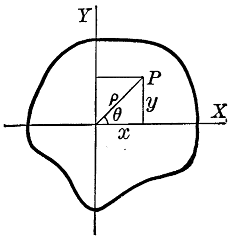
(1) $$\textbf{Moment of area of any surface} = \lim_{\Delta S = 0} \Sigma r \Delta S = \int r dS.$$

In the applications of equation (1) we shall use A for plane areas and S for curved areas only.

The most important case of moment of area of a plane surface is that with respect to axes lying in the plane. Results may always be put in the following standard form:

(2) $$\text{Moment of area} = nSL.$$

The moment of area relative to an axis of symmetry vanishes because of the algebraic sign which must be attached to r, positive on one side of the axis and negative on the other.



20. Calculation of moments of a plane area.

Rectangular coördinates. Using rectangular coördinates in the plane of the area and denoting by C_x and C_y the moments with respect to the axes of X and Y respectively, we have at once by definition,

VIII $$C_x = \int y dA = \iint y dx dy; \quad C_y = \int x dA = \iint x dx dy.$$

Polar coördinates. In polar coördinates we have, approximately, $\Delta A = \rho \Delta \rho \Delta \theta$ (p. 5), and by reference to the figure it is seen that $x = \rho \cos\theta$ and $y = \rho \sin\theta$. Hence the formulæ for moment of area become

IX $$C_x = \iint y \cdot \rho d\rho d\theta = \iint \rho^2 \sin\theta \, d\rho d\theta;$$
$$C_y = \iint x \cdot \rho d\rho d\theta = \iint \rho^2 \cos\theta \, d\rho d\theta.$$

21. Center of area. The center of any plane area is defined by formulæ analogous to those for center of mass, **II**, p. 23. If the plane of the area be chosen as the XY-plane, we have as **formulæ for center of area**

X $$\bar{x} = \frac{C_y}{A} = \frac{\iint x\,dx\,dy}{\iint dx\,dy}; \quad \bar{y} = \frac{C_x}{A} = \frac{\iint y\,dx\,dy}{\iint dx\,dy}.$$

It may easily be shown that an equation similar to (**6**), p. 23, holds for areas. That is, *the perpendicular distance from any line l to the center of area is*

(**1**) $$\bar{r} = \frac{C_l}{A}.$$

In polar coördinates equations **X** become

XI $$\bar{x} = \bar{\rho}\cos\bar{\theta} = \frac{\iint \rho^2 \cos\theta\, d\rho\, d\theta}{\iint \rho\, d\rho\, d\theta}; \quad \bar{y} = \bar{\rho}\sin\bar{\theta} = \frac{\iint \rho^2 \sin\theta\, d\rho\, d\theta}{\iint \rho\, d\rho\, d\theta}.$$

22. Center of mass of thin plates. A thin plate or lamina may be regarded (p. 9) as a cylinder of very small height or thickness h, density τ, and area of base A. The mass of an element of such a solid may be written

$$\Delta m = \tau \cdot h \Delta A.$$

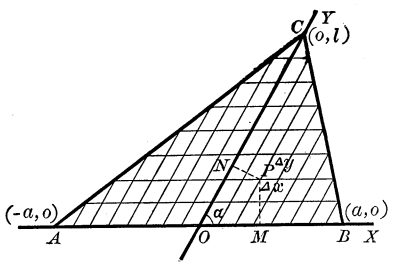
Taking the XY-plane as the plane of the plate, and assuming it to be homogeneous, the formulæ **II**, p. 23, become

$$\bar{x} = \frac{C_{yz}}{M} = \frac{\iint x\,dx\,dy}{\iint dx\,dy}; \qquad \bar{y} = \frac{C_{xz}}{M} = \frac{\iint y\,dx\,dy}{\iint dx\,dy}; \qquad \bar{z} = \frac{h}{2}$$

(by **IV**, p. 24). In the present case $\bar{z}$ is very small, so small in fact that it is usually assumed equal to zero. The values of $\bar{x}$ and $\bar{y}$ are the same as for a plane area. Hence under the given conditions we may say that *for homogeneous thin plates or laminæ centers of mass and centers of area are identical.*

23. Illustrative examples.

1. *Homogeneous plane triangle.* Let there be given any plane triangle ABC and let the base AB be chosen as the axis of X and the median from the vertex opposite the base as the axis of Y, the axes thus chosen making an angle α with each other. Also let $AB = 2a$ and $OC = l$. Then the equation of AC is



$$ay - lx = al \quad \text{or} \quad x = \frac{a(y-l)}{l},$$

and the equation of BC is

$$ay + lx = al \quad \text{or} \quad x = \frac{a(l-y)}{l}.$$

Any element of area is given by $\Delta A = \Delta y \Delta x \sin\alpha$; hence for moment of area relative to the X-axis we have

$$C_x = \int r dA = \iint y \sin\alpha \cdot dy dx \sin\alpha$$
$$= \sin^2\alpha \int_0^l \int_{-\frac{a}{l}(l-y)}^{+\frac{a}{l}(l-y)} y dx dy = \frac{al^2 \sin^2\alpha}{3}.$$

$C_y = 0$ by symmetry. The area of the triangle $= a \cdot l \sin\alpha$, and hence in the standard form C_x may be written

$$C_x = \frac{l \sin\alpha}{3} \cdot al \sin\alpha = \frac{l\sin\alpha}{3} A,$$

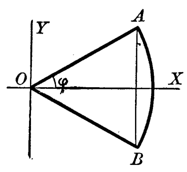
which becomes $C_x = \frac{l}{3} A$ in rectangular coördinates.

In finding the center of area, it must be remembered that the formulæ derived on p. 30 are for rectangular coördinates. The perpendicular distance from the X-axis to the center of area $(0, \bar{y})$ is $\frac{C_x}{A} = \frac{l \sin\alpha}{3}$ (p. 30). Since this distance, in the present example, is also equal to $\bar{y} \sin\alpha$, we have

XII $$\bar{y}\sin\alpha = \frac{l\sin\alpha}{3} \quad \text{or} \quad \bar{y} = \frac{l}{3}.$$

Theorem. *The center of area of a plane triangle, or the center of mass of a homogeneous triangular plate, is situated in the median from the vertex to the base and is one third the length of the median from the base.*

2. *Circular sector.* Let the equation of the circle in polar coördinates be $\rho = a$, and take the line OX bisecting the angle $AOB = 2\phi$ of the sector as the axis of X. For θ, the limits of integration are evidently $-\phi$ and ϕ; for ρ, 0 and a.



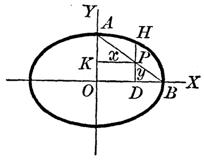
Then, by **IX**, p. 29, the moment of area C_y is

$$C_y = \int_{-\phi}^{+\phi}\int_0^a \rho^2 \cos\theta \, d\rho d\theta = \frac{a^3}{3}\int_{-\phi}^{+\phi} \cos\theta d\theta = \frac{2}{3}a^3 \sin\phi.$$

The area of the sector is $A = \iint \rho d\rho d\theta = a^2\phi,$

and hence the center of area is $\bar{x} = \frac{2}{3} a \frac{\sin\phi}{\phi}$; $\bar{y} = 0$.

3. *Segment of an ellipse.* Let the given segment of the ellipse whose semi-axes are a and b be that cut off by the quadrantal chord AB. The equation of the ellipse is $x^2b^2 + a^2y^2 = a^2b^2$, and the equation of the chord is $ay = b(a - x)$, where $y = DP$ and $x = KP$. Hence the moment of area with respect to y is



$$C_y = \int_0^a \int_{\frac{b}{a}(a-x)}^{\frac{b}{a}\sqrt{a^2-x^2}} xdydx = \int_0^a \left[\frac{b}{a}\sqrt{a^2-x^2} - \frac{b}{a}(a-x)\right] xdx$$

$$= \left[-\frac{b}{3a}(a^2-x^2)^{\frac{3}{2}} - \frac{bx^2}{2} + \frac{bx^3}{3a}\right]_0^a = \frac{a^2b}{6}.$$

Similarly $$C_x = \int_0^a dx \int_{\frac{b}{a}(a-x)}^{\frac{b}{a}\sqrt{a^2-x^2}} ydy = \frac{b^2a}{6}.$$

The area of the segment is

$$A = \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx - \frac{b}{a}\int_0^a (a-x)\,dx = \frac{ab}{4}(\pi - 2).$$

Hence the center of area is at the point

$$\bar{x} = \frac{2a}{3(\pi-2)}; \quad \bar{y} = \frac{2b}{3(\pi-2)}.$$

If problems 1, 2, 3 concerned homogeneous thin plates of the same forms as the areas, the process of solution and the results would be the same as those given above for the areas. The formulæ were shown to be the same for homogeneous plates and for areas on p. 30. In case the plates are not homogeneous they are to be treated as non-homogeneous cylinders.

PROBLEMS

1. A regular hexagonal frame of negligible mass has five equal masses situated at the middle points of five sides. Find the moments and center of mass if the unweighted side coincides with the Y-axis.

2. Find the center of area of the triangle OAB in the circular sector, p. 31.
Ans. $\bar{y} = 0$; $\bar{x} = \frac{2}{3}a\cos\phi$.

3. Find the center of area, (1) of a quarter circle in the first quadrant; (2) of one sixth of a circle, supposing the X-axis to be an axis of symmetry.
Ans. (1) $\bar{x} = \dfrac{4a}{3\pi} = \bar{y}$; (2) $\bar{x} = \dfrac{2a}{\pi}$, $\bar{y} = 0$.

4. Find the center of mass of a quadrant of a homogeneous elliptical plate.
Ans. $\bar{x} = \dfrac{4a}{3\pi}$; $\bar{y} = \dfrac{4b}{3\pi}$.

5. Find the center of the area bounded by $y = a\sin\dfrac{x}{a}$ and the axis of X between $x = 0$ and $x = a\pi$.
Ans. $\bar{x} = \frac{1}{2}a\pi$; $\bar{y} = \frac{1}{8}a\pi$.

6. A thin plate whose density varies as $(h^2 - x^2)^{-\frac{1}{2}}$ is bounded by the lines $y = ax$, $y = 0$, and $x = h$. Find its center of mass. *Ans.* $\bar{x} = \frac{1}{4}\pi h$; $\bar{y} = \frac{1}{8}\pi ah$.

7. Find the center of the area bounded by $y^2 = 4ax$, $y = 0$, and $x = b$.
Ans. $\bar{x} = \frac{3}{5}b$; $\bar{y} = \frac{3}{4}\sqrt{ab}$.

8. Find the center of the area bounded by the hyperbola $xy = c^2$, $x = a$, $x = b$, and $y = 0$.
Ans. $\bar{x} = \dfrac{b-a}{\log b - \log a}$; $\bar{y} = \dfrac{c^2(b-a)}{2ab(\log b - \log a)}$.

24. First theorem of Pappus and Guldinus. Let there be given a plane area bounded by any curve. The center of area is given by equations X, p. 30. Suppose now that this area is revolved about the X-axis; the volume generated is $V = \iint 2\,\pi y dx dy$. The path described by the center of area is a circle whose circumference is

$$(1) \qquad 2\,\pi \bar{y} = \frac{2\,\pi}{A}\iint y dx dy = \frac{V}{A},$$

or *the length of the path described by the center of area is equal to the volume generated, divided by the area in question.*

This proposition is useful for finding the volume of a solid of revolution when the area in question and its center of area are either known or may be found more easily by direct integration than the volume itself. Conversely, if the volume and area are known, the center of the given area may be found.

For example, to find the center of area of a semi-circle having the X-axis as a diameter: if a is the radius of the circle, the area is $\frac{\pi a^2}{2}$ and the volume is $\frac{4}{3}\pi a^3$. Hence

$$2\,\pi \bar{y} = \frac{\frac{4}{3}\pi a^3}{\frac{\pi a^2}{2}} \quad \text{or} \quad \bar{y} = \frac{4\,a}{3\,\pi}.$$

25. Moment and center of area of any curved surface. The general formula for the moment of area of any surface has already been given [(1), p. 29]. For any curved surface referred to rectangular coördinates

$$\Delta S = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\Delta x \Delta y,$$

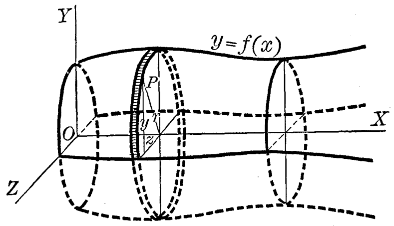
which being substituted in equation (1), p. 29, gives

$$C_E = \iint_S r\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dx dy.$$

The center of area is evidently given by $\bar{x} = \frac{C_E}{S}$, etc.

For curved surfaces the important moments are those with respect to planes.

26. Surfaces of revolution. Let the axis of X be the axis of revolution and the generating curve $y = f(x)$. Also let Δs be an element of arc of the generating curve. Then during the revolution of $y = f(x)$ about X the element of arc Δs will generate a small strip or elementary area $\Delta S'$ of the surface, which is approximately the surface of a frustum of a cone of revolution. Since the slant height is Δs and the radius of the median section is y, we have $\Delta S' = 2\pi y \Delta s$, which is more nearly correct the smaller Δs is taken. The elementary moment of area with respect to YZ is approximately*



$$x\Delta S' = 2\pi y x \Delta s,$$

[Since x is approximately the distance from *any* point of ΔS to YZ.]

and the moment of area of the entire surface is the limit of the sum of all such moments; that is,

$$\text{Moment of area} = \text{limit}\ \Sigma x\Delta S' = 2\pi \lim_{\Delta s = 0} \Sigma xy\Delta s,$$

or

$$C_{yz} = \int x dS' = 2\pi \int xyds.$$

Theorem. *The moment of area of the surface of any solid of revolution with respect to a plane YZ perpendicular to its axis is given by*

$$\text{XIII} \qquad \boldsymbol{C_{yz} = 2\pi \int xyds = 2\pi \int \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} xydx,}$$

where OX is the axis of revolution and $y = f(x)$ is the equation of the generating curve.

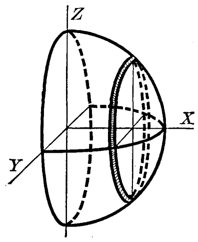
* The value of C_{yz} may be found directly from the general definition (p. 33) as follows. By definition we have approximately,

$$C_{yz} = \Sigma\Sigma x\Delta S = \Sigma\Sigma x\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} \Delta x \Delta y,$$

the double summation indicating that both x and y vary. Calculating this sum for any constant value of x, we have $C_{yz} = \Sigma x \cdot \Sigma\Delta S$. But $\Sigma\Delta S$ is the area of the strip $\Sigma\Delta S = \Delta S' = 2\pi y \Delta s$ generated by the element of arc ΔS as the curve $y = f(x)$ is revolved about OX; that is, $\Sigma\Delta S = 2\pi y \Delta s$. Hence approximately $C_{yz} = \Sigma x \cdot 2\pi y \Delta s$, as above.

Illustrative example. *Find the moment and center of area of the surface of a hemisphere.*

Let the equation of the generating circle be $x^2+y^2=a^2$. Consider the hemisphere to the right of the YZ-plane. Then $\frac{dy}{dx}=-\frac{x}{y}$, and from **XIII,**



$$C_{yz}=\int xdS=2\pi\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,xydx=2\pi a\int xdx=\pi a^3.$$

The center of area is then at the point $\bar{y}=0$

and $$\bar{x}=\frac{C_{yz}}{S}=\frac{\pi a^3}{2\pi a^2}=\frac{a}{2}.$$

It should be noted that in this case the center of area is a point outside the surface itself, and this is usually the case with arcs and curved surfaces.

Exactly the same relations hold between the formulæ for curved surfaces and thin shells as between those for plane areas and thin plates, p. 30.

PROBLEMS

1. Find the moment and center of area of the surface of the cone formed by the revolution about the X-axis of the segment of the line $hy=ax$ included between the origin and $x=h$. *Ans.* $C_{yz}=\frac{2}{3}\pi ah\sqrt{h^2+a^2}$; $\bar{x}=\frac{2}{3}h$.

2. Find the center and moment of area of the surface formed by the revolution of a quadrant of a circumference about a tangent at its extremity.

Ans. $\bar{y}=\frac{a}{\pi-2}$.

3. Find the center of mass of the first octant of a thin spherical shell whose density varies as z, the origin being the center. *Ans.* $\bar{x}=\bar{y}=\frac{4a}{3\pi}$; $\bar{z}=\frac{2a}{3}$.

4. Find the center of mass of a thin homogeneous shell which is that portion of the surface of a right circular cylinder $x^2+z^2=a^2$, cut out by the planes $y=x\tan\phi$, $z=0$, and $x=0$.

5. Find by the theorem of Pappus (1) the center of area of one fourth of a circle in the first quadrant; (2) the volume generated by revolving the circle $(x-a)^2+y^2=r^2$ about the axis of Y. *Ans.* (1) $\bar{x}=\bar{y}=\frac{4a}{3\pi}$; (2) $V=2\pi^2ar^2$.

27. Moment and center of arc. The definition of moment of arc should be evident from the preceding definitions of moment of mass and moment of area. Let a given curve be divided into elements of arc Δs, and let r represent the distance from a given point, line, or plane to a point suitably chosen in each element. Then

(1) $$\textbf{Moment of arc} = \lim_{\Delta s=0}\Sigma r\Delta s=\int r\,ds.$$

Results may always be put in the standard form $= nsL$, where $s =$ total length of arc and n and L are as before.

The center of arc is defined in exactly the same way as center of area and center of mass.

In rectangular coördinates the moments of arc for a plane curve are

XIV $$C_x = \int y\,ds; \qquad C_y = \int x\,ds.$$

The **center of arc** is given by

XV $$\bar{x} = \frac{\int x\,ds}{\int ds}; \qquad \bar{y} = \frac{\int y\,ds}{\int ds};$$

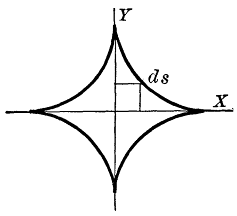
where ds may be expressed either in rectangular or in polar coördinates.

If we consider a *material curve* such as a thin wire,

$$C_x = \int \tau y\,ds; \qquad \bar{x} = \frac{\int \tau x\,ds}{\int \tau\,ds}, \text{ etc.}$$

Illustrative example. *To find the moments and center of arc of a quadrant of the hypocycloid* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Consider the part of the curve lying in the first quadrant. The limits of integration are 0 and a. Hence



$$C_y = \int_0^a x\,ds = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,x\,dx$$

$$= \int_0^a \sqrt{\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\,x\,dx = a^{\frac{1}{3}} \int_0^a x^{\frac{2}{3}}dx = \frac{3}{5}a^2.$$

Similarly $C_x = \frac{3}{5}a^2$. Also $s = \int ds = \frac{3}{2}a$.

Hence the center of arc is $\bar{x} = \bar{y} = \frac{2}{5}a$.

28. Second theorem of Pappus and Guldinus. Let there be given any plane curve, say in the XY-plane. An element of the curve is approximately $\Delta s = (\Delta x^2 + \Delta y^2)^{\frac{1}{2}}$. Now suppose the curve to revolve about the X-axis. Its center of arc will describe the circle $2\pi\bar{y}$, which from equations (**XV**) is equivalent to

(1) $$2\pi\bar{y} = \frac{2\pi}{s}\int y\,ds = \frac{2\pi}{s}\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,y\,dx.$$

But the numerator of the right-hand member of this equation is the surface of revolution generated by the curve as it revolves about OX. Hence

$$(2)\qquad 2\,\pi\bar{y} = \frac{S}{s}\cdot$$

That is, *the length of the path described by the center of arc of the curve is equal to the area of the surface generated, divided by the length of the curve.* Evidently a similar relation holds for $\bar{x}$ when the curve is revolved about the Y-axis.

Hence, if two of the quantities S, s, and $\bar{y}$ are known or may easily be found by direct methods, the relation (2) enables the third to be calculated very simply.

29. Principle of combination. If a solid consists of two or more parts, we may, so far as the moment of mass is concerned, regard the mass of each part as concentrated at its center of mass (p. 22). The center of mass of the entire figure is found by considering the several centers of mass of the parts as a system of particles whose masses are the masses of the corresponding portions of the solid. If a portion is cut away, the mass of that part is considered negative.

Suppose a solid S of mass M is composed of two parts, S_1 and S_2, of masses m_1 and m_2 respectively. Also let $\bar{P}(\bar{x},\bar{y},\bar{z})$, $\bar{P}_1(\bar{x}_1,\bar{y}_1,\bar{z}_1)$, and $\bar{P}_2(\bar{x}_2, \bar{y}_2, \bar{z}_2)$ be the respective centers of mass of the solids S, S_1, and S_2. Evidently the moment of mass of S with respect to any plane equals the sum of the moments of mass of S_1 and S_2 with respect to the same plane. Taking moments with respect to the coördinate planes, therefore, we have

$$(1)\qquad \begin{aligned} M\bar{x} &= m_1\bar{x}_1 + m_2\bar{x}_2,\\ M\bar{y} &= m_1\bar{y}_1 + m_2\bar{y}_2,\\ M\bar{z} &= m_1\bar{z}_1 + m_2\bar{z}_2. \end{aligned}$$

Since $M = m_1 + m_2$ we find, by solving,

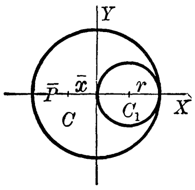
$$(2)\qquad \bar{x} = \frac{m_1\bar{x}_1 + m_2\bar{x}_2}{m_1 + m_2};\quad \bar{y} = \frac{m_1\bar{y}_1 + m_2\bar{y}_2}{m_1 + m_2};\quad \bar{z} = \frac{m_1\bar{z}_1 + m_2\bar{z}_2}{m_1 + m_2}.$$

By comparing with Analytic Geometry, p. 335, it will be seen that equations (2) are identical with those defining the coördinates of a point which divides a given line in the ratio $\lambda = \dfrac{m_2}{m_1}$.

Theorem. *If $\bar{P}_1$ and $\bar{P}_2$ are the centers of mass of two solids S_1 and S_2, of masses m_1 and m_2, the center of mass $\bar{P}$ of the solid formed by combining S_1 and S_2 lies on the line joining $\bar{P}_1$ and $\bar{P}_2$ and divides it in the ratio*

$$\frac{\bar{P}_1\bar{P}}{\bar{P}\bar{P}_2} = \frac{m_2}{m_1}.$$

Illustrative example. It is required to find the center of mass of the remainder of a circular plate of radius $2r$ after a plate of radius r has been removed as indicated in the figure. Assuming the plate to be homogeneous, the mass of the part removed is one fourth the total mass m_2 and is to be taken as negative. Let $\bar{P}$ be the center of mass sought. Then we have



$$m_1 = -\tau\pi r^2,$$
$$m_2 = 4\tau\pi r^2,$$
$$M = \tau\pi(4r^2 - r^2) = 3\tau\pi r^2.$$

Substituting in equations (**2**), we obtain

$$\bar{x} = \frac{-\tau\pi r^2 \cdot r + m_2 \cdot 0}{3\tau\pi r^2} = -\frac{r}{3}.$$

Both $\bar{y}$ and $\bar{z}$ are zero by symmetry. Hence the center of mass lies on the axis of X, $\frac{1}{3}r$ in the negative direction from the origin.

PROBLEMS

In the following list of problems the results for moments of mass, area, and arc are not given, although they must of necessity be found since they occur as numerators in the expressions for centers of mass, area, etc.

I. ARCS

1. Find the center of arc of a sector of the circle $\rho = a$ between $-\theta$ and $+\theta$, and from this derive the results for quadrantal and semi-circular arcs.

Ans. $\bar{x} = \dfrac{a \sin\theta}{\theta}$. For quadrantal arc $\theta = \dfrac{\pi}{4}$ and $\bar{x} = \dfrac{4a}{\pi\sqrt{2}}$.

For semi-circular arc $\theta = \dfrac{\pi}{2}$ and $\bar{x} = \dfrac{2a}{\pi}$.

2. Find the center of mass of a thin straight wire of length a whose density varies as the nth power of the distance from one end. *Ans.* $\bar{x} = \dfrac{n+1}{n+2}a$.

3. Find the center of arc of the perimeter of the cardioid $\rho = a(1 - \cos\theta)$.

Ans. $\bar{x} = -\frac{4}{5}a$.

4. Find the center of arc of the cycloid $x = a \operatorname{vers}^{-1}\dfrac{y}{a} - (2ay - y^2)^{\frac{1}{2}}$ between two successive cusps.

Ans. $\bar{x} = a\pi$; $\bar{y} = \dfrac{4a}{3}$.

5. A wire whose thickness varies as the distance from the middle point is bent into the form of a cycloid. Find the center of mass, taking the origin at the middle point of the wire and the axis of the cycloid as the X-axis. *Ans.* $\bar{y} = 0$; $\bar{x} = a$.

6. Find the center of arc of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ between $x = a$ and $x = -a$.

Ans. $\bar{y} = \dfrac{e^4 + 4e^2 - 1}{4e(e^2 - 1)}$.

7. Find the center of arc of the helix $x = a\cos\theta$; $y = a\sin\theta$; $z = ka\theta$ from the origin to $\theta = \theta_1$.

Ans. $\bar{x} = \dfrac{a\sin\theta}{\theta}$; $\bar{y} = \dfrac{a(1-\cos\theta)}{\theta}$; $\bar{z} = \dfrac{ka\theta}{2}$.

8. Find the center of the arc of the parabola $y^2 = 4ax$ included between the origin and the point where it is cut by an ordinate through the focus.

Ans. $\bar{x} = \dfrac{a}{4}\,\dfrac{3\sqrt{2} - \log(1+\sqrt{2})}{\sqrt{2} + \log(1+\sqrt{2})}$; $\bar{y} = \dfrac{4a}{3}\,\dfrac{2\sqrt{2} - 1}{\sqrt{2} + \log(1+\sqrt{2})}$.

II. PLANE AREAS

1. Find the center of mass and the mean density of a rectangular plate whose sides are $2a$ and $2b$, if the density varies as the distance from the side $2b$.

[Let the axes coincide with the sides of the plate.]

Ans. $\bar{x} = \dfrac{4a}{3}$; $\bar{y} = b$; $\bar{\tau} = ka$.

2. Find the center of mass of the first quadrant of a circular plate whose density varies as xy. *Ans.* $\bar{x} = \bar{y} = \frac{8}{15}a$.

3. Find the center of the area included by $ax^n = \pm y$, $x = b$, and $x = c$, where $c > b$.

Ans. $\bar{x} = \dfrac{c^{n+2} - b^{n+2}}{c^{n+1} - b^{n+1}}\,\dfrac{n+1}{n+2}$.

4. Find the center of the area between the parabola $y^2 = 4ax$, the Y-axis, and $y = b$.

5. Find the center of the area bounded by the semi-cubical parabola $ay^2 = x^3$ and $x = a$. *Ans.* $\bar{x} = \frac{5}{7}a$.

6. Find the distance of the center of area from the vertices of a right triangle whose sides are a and b. *Ans.* $\frac{1}{3}\sqrt{a^2 + b^2}$; $\frac{1}{3}\sqrt{4a^2 + b^2}$; $\frac{1}{3}\sqrt{a^2 + 4b^2}$.

7. A given equilateral triangle is cut out of a circle. Find the center of the remaining area, supposing the center of the triangle to be situated at the distance b from the center of the circle.

8. If five ninths be cut away from a triangle by a line parallel to the base, show that the center of the remaining area divides the median in the ratio of 4 : 5.

9. One corner of a square plate is cut off by a line joining the middle points of two adjacent sides. Find the center of mass of the remainder.

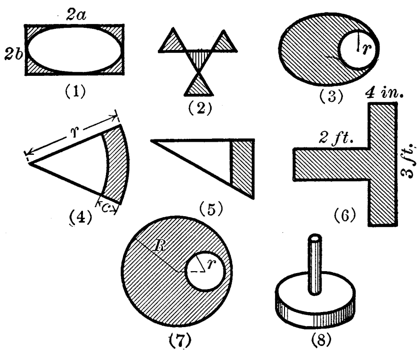
On the diagonal from the corner cut off at a distance $\dfrac{a\sqrt{2}}{21}$ from the intersection of the diagonals.

10. Find the center of area (1) of a loop of the curve $\rho = a\cos 2\theta$; (2) of the curve $\rho^2 = a^2\cos 2\theta$.

Ans. (1) $\bar{x} = \dfrac{128}{105}\,\dfrac{a\sqrt{2}}{\pi}$; (2) $\bar{x} = \dfrac{\pi a\sqrt{2}}{8}$.

11. An equilateral triangle is formed on one side of a square. Find the center of the whole area. *Ans.* $\dfrac{3\,a}{8+2\sqrt{3}}$ from the base of the triangle.

12. Find the centers of area of the shaded portions of the following figures.



13. One corner of a square of side $2\,a$ is cut off by a line drawn from a corner to the middle point of an opposite side. The opposite corner is also cut off by removing a circle of radius ρ having its center at the corner. Find center of area of the remainder.

14. Find the center of mass of the homogeneous thin plate bounded by the curves $y^2 = ax$ and $x^2 = by$.

Ans. $\bar{x} = \frac{9}{20}\,a^{\frac{1}{3}}b^{\frac{2}{3}}$; $\bar{y} = \frac{9}{20}\,a^{\frac{2}{3}}b^{\frac{1}{3}}$.

15. Find the center of the area included between the coördinate axes and the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

16. Find the center of the area of the cardioid $\rho = 2\,a\,(1 - \cos\theta)$. Explain the negative sign. *Ans.* $\bar{x} = -\frac{5}{3}\,a$.

17. Find the center of the area of the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$.
Ans. $\bar{x} = a\pi$; $\bar{y} = \frac{5}{6}\,a$.

18. Find the center of the area of the parabola $y^2 = 4\,ax$ cut off by the line $3\,y - 2\,x = 4\,a$. *Ans.* $\bar{x} = \frac{12}{5}\,a$; $\bar{y} = 3\,a$.

19. Find the center of the area of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
Ans. $\bar{x} = \bar{y} = \dfrac{256}{315}\dfrac{a}{\pi}$.

20. Find the center of area of the segment of the circle $x^2 + y^2 = a^2$ cut off by the ordinate $x = a\cos\phi$. *Ans.* $\bar{x} = \dfrac{2\,a\sin^3\phi}{3\,(\phi - \sin\phi\cos\phi)}$.

21. Find the center of the area included between the first and second spires of the spiral of Archimedes $\rho = a\theta$.

III. SOLIDS AND CURVED SURFACES

1. A cylinder is 12 in. long and for 8 in. of its length has a diameter of 4 in.; for the remaining 4 in. it has a diameter of 3 in. Find center of mass.
Ans. $5\frac{13}{41}$ in. from thick end.

2. Find the center of area of the surface of a hemisphere where the density of each point on the surface varies as its distance from the base of the hemisphere.
Ans. $\bar{x} = \frac{2}{3}\,a$.

3. Use the theorems of Pappus to prove that the volume and the surface generated by the revolution of a cycloid about a tangent at its vertex are respectively $\pi^2 a^3$ and $\frac{32}{3}\,\pi a^2$.

4. Find the center of area of the surface formed by revolving the arc of the parabola $y^2 = 4\,ax$ between the origin and the point $(a,\ 2a)$ about the Y-axis.

5. Find the center of mass of the solid formed in Ex. 4. *Ans.* $\bar{y} = \frac{5}{8}\,a.$

6. Show that the center of mass of the solid formed by revolving the area included by the arc in Ex. 4, the X-axis, and $x = a$, about $x = a$, is $\bar{y} = \frac{5}{8}\,a.$

7. Find the center of mass of the paraboloid of revolution whose generating curve is $y^2 = 4\,ax$, between the origin and $x = b$. *Ans.* $\bar{x} = \frac{2}{3}\,b.$

8. Find the center of area of the surface formed in Ex. 7.

9. A cone having the same base and vertex is cut from the paraboloid of revolution in Ex. 7. Find the center of mass of the remaining solid. *Ans.* $\bar{x} = \dfrac{b}{2}.$

10. Find the center of mass of the solid formed by the revolution of the curve $y^4 - axy^2 + x^4 = 0$ about the axis of X. *Ans.* $\bar{x} = \dfrac{3\,a\pi}{32}.$

11. Find the center of mass of the solid included by $c^2z^2 = y^2\,(a^2 - x^2)$ and the planes $z = 0$, $y = c$, and $x = 0$. *Ans.* $\bar{x} = \dfrac{4\,a}{3\pi}$; $\bar{y} = \dfrac{2}{3}\,c$; $\bar{z} = \dfrac{8\,a}{9\pi}.$

12. Find the center of mass of the solid and the center of area of the surface formed by the revolution of the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$ about its base.
Ans. For volume, $\bar{x} = \dfrac{45\,\pi^2 + 128\,a}{15}$; for surface, $\bar{x} = \dfrac{26}{15}\,a.$

13. Find the centers of mass and area of the surface of the solid generated by revolving the cycloid about its axis.

Ans. For volume, $\bar{x} = \dfrac{63\,\pi^2 - 64}{6\,(9\,\pi^2 - 16)}\,a$; for surface, $\bar{x} = \dfrac{2\,(15\,\pi - 8)}{15\,(3\,\pi - 4)}\,a.$

14. The cardioid $\rho = 2\,a\,(1 - \cos\theta)$ revolves about its axis. Find the center of area of the surface formed. *Ans.* $\bar{x} = -\,\frac{100}{63}\,a.$

15. A right circular cone of height h and radius of base a is cut by two planes through its axis inclined at an angle θ to each other. Find the center of area of the surface cut out.

[Take planes $y = 0$ and $z = y\tan\theta$ and axes as on p. 57.]

Ans. $\bar{x} = \dfrac{2}{3}\,h$; $\bar{y} = \dfrac{2}{3}\,a\,\dfrac{\sin\theta}{\theta}$; $\bar{z} = \dfrac{2}{3}\,a\,\dfrac{1 - \cos\theta}{\theta}.$

16. The axes of two cylinders each of radius a intersect perpendicularly. Find the center of mass of the volume included by the two cylinders and a plane through their axes. *Ans.* $\frac{3}{8}\,a$ from plane through axes.

17. Find the center of mass of the solid formed by the revolution of a sector of a circle about one of its extreme radii, the angle between the radii being β.
Ans. $\bar{x} = \frac{3}{4}\,a\cos^2\frac{1}{2}\,\beta.$

18. From a sphere of radius R is removed a sphere of radius r, the distance between their centers being c. Find the center of mass of the remainder.

Ans. On the line joining the centers at a distance $\dfrac{cr^3}{R^3 - r^3}$ from the center of the larger sphere.

19. A bowl in the form of a hemisphere is closed by a flat circular lid of a material whose density is three times that of the bowl. Find the center of mass of the whole, neglecting the thickness of the material. *Ans.* $\frac{1}{5}r$ from center.

20. Find the center of mass of the solid included by the coördinate planes and the hyperbolic paraboloid $y = b\left(\frac{a-x}{a}\right)\left(\frac{c-x}{c}\right)$. *Ans.* $\bar{x} = \frac{a}{3}$; $\bar{y} = \frac{2}{9}b$; $\bar{z} = \frac{c}{3}$.

21. Find the center of mass of the solid of revolution about the Z-axis, where the area of a section perpendicular to z varies (1) as $(a-z)$; (2) as $(a-z)^3$; (3) as $\sqrt{a-z^2}$; (4) as $\sin(a-z)$.

22. Find the center of area of the surface of a frustum of the cone formed by the revolution of the line $hy = mx$ about the axis of X between $x = a$ and $x = b$, where $a > b$.

$$Ans.\quad \bar{x} = \frac{2(a^2+ab+b^2)}{3(a+b)}.$$

23. Find the center of mass of the solid formed in Ex. 22.

$$Ans.\quad \bar{x} = \frac{3(a+b)(a^2+b^2)}{4(a^2+ab+b^2)}.$$

CHAPTER III

MOMENT OF INERTIA. $\int r^2 dm$

30. Moment of inertia.* *The moment of inertia of a material particle with respect to a plane is the product of the mass of the particle and the square of its distance from the plane.*

Thus if m is the mass of a particle and r is its perpendicular distance from a fixed plane E, the moment of inertia of the particle with respect to the plane is r^2m. Again, suppose there is a system of particles of masses $m_1, \cdots, m_n$ situated at distances $r_1, \cdots, r_n$ respectively from the plane of reference E; then the sum of the moments of inertia, namely

$$\Sigma r^2 m, \tag{1}$$

is called *the moment of inertia of the system with respect to E.*

The idea of moment of inertia may be extended to a continuous solid S precisely as was the similar conception of moment of mass, p. 18, using r^2 instead of r. Hence we have at once, by analogy: *the moment of inertia of a continuous solid with respect to a fixed plane E is defined by*

$$\underset{\Delta m = 0}{\text{limit}}\ \Sigma r^2 \Delta m = \int r^2 dm. \tag{2}$$

This integral† is denoted by I. Hence with respect to a fixed plane E,

$$\textbf{Moment of inertia} = \boldsymbol{I_E} = \int \boldsymbol{r^2 dm}, \tag{3}$$

where r is the perpendicular distance from any point of the solid to the plane E.

* Moment of inertia is sometimes called *moment of the second order*, since the distance enters as the second power.

† In taking the limit in (2) when Δm approaches zero, it is usual to choose Δm so that *all* the dimensions of the element of mass approach zero simultaneously. This is, however, not a necessary restriction.

It should be noted that moment of inertia is never negative. Mass is essentially positive, and r, which may be negative, always enters as the square; consequently symmetry does not play the same rôle in calculating moments of inertia that it does in finding moment of mass (p. 19). The moment of inertia of a body is never zero.

31. Standard form of moment of inertia. Since the moment of inertia of a single particle is the product of the square of a distance by a mass, the moment of inertia of a system of particles or of a solid will necessarily be of the same form, with the addition of a numerical factor. That is, we may always express moment of inertia in the standard form,

(1) $$I_E = nL^2M.$$

32. Radius of inertia. A single particle whose mass M equals that of a given solid may evidently be placed at a distance r_0 from the plane of reference E, so that

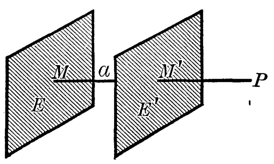
(1) $$I_E = \int r^2 dm = Mr_0^2.$$

Then the moments of inertia of the particle and of the solid with respect to the plane of reference are equal. This distance r_0 is called the *radius of inertia* or the *radius of gyration* of the solid for the plane of reference E. Obviously r_0 is different, in general, for different planes of reference. By solving (1) we find

I $$r_0^2 = \frac{I_E}{M}.$$

Theorem. *The square of the radius of inertia of any solid with respect to a given plane is equal to its moment of inertia with respect to that plane divided by its mass.*

33. Moment of inertia with respect to parallel planes. Let P be any point in a solid S, E and E' two parallel planes, and $MM' = a$ the distance between them. Then if $MP = r$ and $M'P = r'$, we have $r = a + r'$, and hence



$$I_E = \int r^2 dm = \int (a + r')^2 dm = \int r'^2 dm + 2\,a \int r' dm + a^2 \int dm.$$

But $\int dm = M$; $\int r'dm = C_{E'}$ (p. 19); and $\int r'^2 dm = I_{E'}$.

(1) $$\therefore\ I_E = I_{E'} + a^2M + 2\,aC_{E'}.$$

If, however, E' is a centroidal plane, $C_{E'} = 0$ (p. 19) and we have the important result

II $$\boldsymbol{I_E = I_{E'} + a^2M.}$$

Theorem. *The moment of inertia with respect to any plane is equal to the moment of inertia with respect to a parallel plane through the center of mass, increased by the product of the entire mass and the square of the distance between the planes.*

Since in any set of parallel planes one and only one passes through the center of mass, it follows at once from **II** that *of all moments of inertia with respect to parallel planes that with respect to the centroidal plane is the least.*

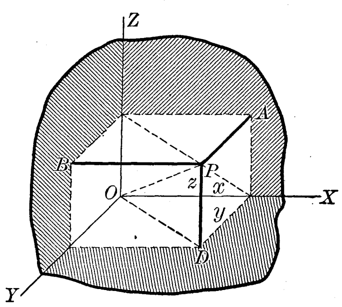
34. Moment of inertia with respect to an axis or a point. The moment of inertia with respect to a line or a point is defined in the same way as with respect to a plane, the distance from any point of the solid to a fixed plane being replaced by its distance to an axis or to a point as the case may be. The moment of inertia with respect to a line l is

(1) $$I_l = \int r^2 dm;$$

and with respect to a point P,

(2) $$I_P = \int r^2 dm,$$

the difference between the formulæ (3), p. 43, and (1) and (2) above being in the interpretation of r. Moments of inertia with respect to points, lines, and planes are equally important and useful.



35. Calculation of moments of inertia for a solid. The preceding definitions enable us to write at once the formulæ for the calculation of moments of inertia.

According to the notation used above

I_{yz} = Moment of inertia with respect to the plane YZ, etc.

I_x = Moment of inertia with respect to the axis OX, etc.

I_o = Moment of inertia with respect to the origin O. Hence the **moments of inertia of a solid with respect to the coördinate planes are**

III $$I_{yz} = \iiint x^2 \cdot \tau dz dy dx; \quad I_{xz} = \iiint y^2 \cdot \tau dz dy dx;$$
$$I_{xy} = \iiint z^2 \cdot \tau dz dy dx,$$

the distance r in the general formula (3), p. 43, becoming x when the plane of reference is YZ, etc., and dm being expressed in rectangular coördinates.

Similarly the **moments of inertia of a solid with respect to the coördinate axes are**

IV $$I_x = \iiint (y^2 + z^2) \cdot \tau dz dy dx; \; I_y = \iiint (x^2 + z^2) \cdot \tau dz dy dx;$$
$$I_z = \iiint (x^2 + y^2) \cdot \tau dz dy dx,$$

where $r^2 = y^2 + z^2$ when the axis of reference is OX, etc.

The **moment of inertia of a solid with respect to the origin is**

V $$I_O = \iiint (x^2 + y^2 + z^2) \cdot \tau dz dy dx.$$

By comparing equations **IV** and **V** we find

$$I_O = \tfrac{1}{2} \iiint 2\, \tau (x^2 + y^2 + z^2)\, dz dy dx$$
$$= \tfrac{1}{2} \iiint \tau [(y^2 + z^2) + (x^2 + z^2) + (x^2 + y^2)]\, dz dy dx.$$

VI $$\therefore \; I_O = \tfrac{1}{2}(I_x + I_y + I_z).$$

Theorem. *The moment of inertia of any solid with respect to a point is equal to one half the sum of the moments of inertia with respect to three perpendicular axes through the point.*

From **VI** it may be observed further that the sum of the moments of inertia with respect to three perpendicular axes through a point is always the same, since that sum is twice the moment of inertia with respect to the point which remains fixed.

A comparison of equations **III** and **V** shows that the following relation also holds.

VII $$\boldsymbol{I_O = I_{xy} + I_{yz} + I_{xz}}.$$

Theorem. *The moment of inertia of any solid with respect to a point is equal to the sum of the moments of inertia with respect to any three perpendicular planes through the point.*

Likewise, from **III** and **IV**,

VIII $$\begin{cases} \boldsymbol{I_x = I_{xy} + I_{xz}}, \\ \boldsymbol{I_y = I_{xy} + I_{yz}}, \\ \boldsymbol{I_z = I_{yz} + I_{xz}}. \end{cases}$$

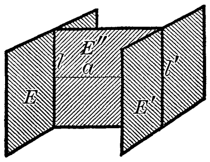
Theorem. *The moment of inertia of a solid with respect to an axis is equal to the sum of the moments of inertia with respect to any two perpendicular planes through the axis.*

Again, from **III**, **IV**, and **V** we find

IX $$\boldsymbol{I_O = I_{xy} + I_z}.$$

Theorem. *The moment of inertia of any solid with respect to a point is equal to the sum of the moments of inertia with respect to a plane through the point and with respect to a line through the point perpendicular to the plane.*

36. Moments of inertia with respect to parallel axes. Let l and l' be any two parallel lines. Let E'' be the plane passed through the two lines l and l', and let E and E' be planes through l and l' respectively perpendicular to E''. Then, from (**1**), p. 45,



$$I_E = I_{E'} + Ma^2 + 2\,aC_{E'}.$$

Adding $I_{E''}$ to both members and applying equations **VIII**,

$$I_l = I_{l'} + Ma^2 + 2\,aC_{E'}.$$

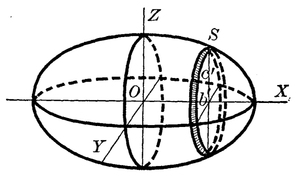
But if E' is a centroidal plane, $C_{E'} = 0$ and

X $$\boldsymbol{I_l = I_{l'} + Ma^2}.$$

Theorem. *The moment of inertia with respect to any line is equal to the moment of inertia with respect to the parallel line through the center of mass plus the product of the total mass into the square of the distance between the lines.*

Of course it is possible in each case to find the moment of inertia by integrating directly between the proper limits. But where the object is not merely an exercise in integration it is much easier to find the moment of inertia with respect to those planes or axes for which the integrations take the simplest possible forms, and then to calculate it for other planes and axes by means of the relations established in the preceding paragraphs. In general, the simplest procedure is to choose three perpendicular planes through the center of mass, if the center of mass is known. When I is calculated for each of these three planes, it is easily found for parallel planes, for the coördinate axes, and for lines parallel to the axes, without further integration.

Illustrative example. *It is required to find the moments of inertia of any homogeneous ellipsoid.* The equation of the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and the area of any section S parallel to the plane YZ is (p. 7)



(1) $$\iint dydz = \pi b'c' = \frac{\pi bc}{a^2}(a^2-x^2).$$

By **III**, p. 46, since τ is constant,

$$I_{yz} = \tau\iiint x^2 dzdydx = \tau\int x^2dx \cdot \iint dydz.$$

(2) $$I_{yz} = \tau\int_{-a}^{+a} x^2dx \cdot \frac{\pi bc}{a^2}(a^2-x^2) = \frac{4}{15}\tau\pi a^3bc.$$

Since the volume of the ellipsoid is $V=\frac{4}{3}\pi abc$ and $M=\tau V$, we may express I_{yz} in the standard form (p. 44) as $I_{yz}=\frac{Ma^2}{5}$. In the same way $I_{xz}=\frac{Mb^2}{5}$ and $I_{xy}=\frac{Mc^2}{5}$.

For the moment of inertia with respect to the X-axis we have, by **VIII**, p. 47,

(3) $$I_x = I_{xy}+I_{xz} = \tfrac{1}{5}M(b^2+c^2).$$

I_y and I_z are found in the same way.

By **VII**, p. 47, the moment of inertia with respect to the origin is easily shown to be

(4) $$I_O = \tfrac{1}{5}M(a^2+b^2+c^2).$$

The moment of inertia with respect to a line l through the extremity of the major axis parallel to the Z-axis is, by **X**, p. 47,

(5) $$I_l = I_z + Ma^2 = \tfrac{1}{5}M(a^2+b^2)+Ma^2.$$

The moment of inertia with respect to a plane E tangent to the ellipsoid at the extremity of the major axis is, by **II**, p. 47,

(6) $$I_E = I_{yz}+Ma^2 = \tfrac{1}{5}Ma^2+Ma^2 = \tfrac{6}{5}Ma^2.$$

PROBLEMS

1. Three edges of a unit cubical frame without weight are taken as the coördinate axes, and particles are placed at all the corners except at the origin. Find I and r_0, (1) when the particles are of equal mass; (2) when the masses vary as the squares of their distances from the origin.

2. Find I and r_0^2 (1) for a rectangular parallelopiped of edges a, b, c, about a line through the centroid parallel to the edge a; (2) for a plane perpendicular to the edge a at a distance d from the center of the parallelopiped.

$$\textit{Ans.}\quad (1)\begin{cases} I = \dfrac{M(b^2+c^2)}{12}; \\ r_0^2 = \dfrac{b^2+c^2}{12}; \end{cases} \quad (2)\begin{cases} I = M(\frac{1}{12}a^2 + d^2). \\ r_0^2 = \frac{1}{12}a^2 + d^2. \end{cases}$$

3. Find I for a rectangular parallelopiped about (1) an edge a; (2) about its center of mass.

$$\textit{Ans.}\quad (1)\ I = \frac{M}{3}(b^2+c^2);\quad (2)\ I = \frac{M}{12}(a^2+b^2+c^2).$$

4. Find r_0^2 for an ellipsoid relative to the coördinate planes and axes taken through the center of mass.

$$\textit{Ans.}\quad \text{For planes, } r_0^2 = \frac{a^2}{5},\ \frac{b^2}{5},\ \frac{c^2}{5};\ \text{for axes, } r_0^2 = \frac{b^2+c^2}{5},\ \text{etc.}$$

5. Find I and r_0^2 for a solid sphere with reference to a diametral plane.

$$\textit{Ans.}\quad I = \frac{Ma^2}{5};\ r_0^2 = \frac{a^2}{5}.$$

6. Find I for a sphere (1) relative to a diameter; (2) relative to the center.

Ans. (1) $\frac{2}{5}Mr^2$; (2) $\frac{3}{5}Mr^2$.

7. Find I and r_0 for a sphere (1) relative to a tangent line; (2) relative to a tangent plane.

Ans. (1) $I = \frac{7}{5}a^2M$; (2) $I = \frac{6}{5}Ma^2$.

37. Moment of inertia of areas and arcs. The moment of inertia of an area is defined in a manner precisely analogous to that of a solid, the element of area replacing the element of mass. Hence for either a plane or a curved surface*

$$(1)\qquad \textbf{Moment of inertia of an area} = I = \lim_{\Delta S=0} \Sigma r^2\Delta S = \int r^2 dS.$$

Similarly

$$(2)\qquad \textbf{Moment of inertia of an arc} = I = \lim_{\Delta s=0} \Sigma r^2\Delta s = \int r^2 ds.$$

* In (1) and (2) it must be remembered that r is the distance from any point P of the area or arc to the plane, line, or point of reference. Also ΔS and Δs must be so chosen that in passing to the limit each element shrinks up into the point P which has been chosen in that element (p. 18).

Results in any particular case may evidently be put in the standard form,

(3) $$I = nL^2S \text{ for areas,}$$

(4) $$I = nL^2s \text{ for arcs.}$$

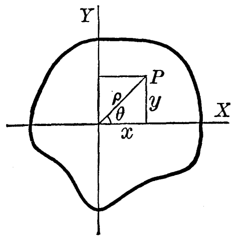
The radius of inertia for an area is defined by

(5) $$r_0^2 = \frac{I}{S},$$

which should be compared with the similar equation **I**, p. 44, defining radius of inertia for a solid.

38. Calculation of moments of inertia of plane areas.

Rectangular coördinates. For a plane area referred to rectangular coördinates in that plane the definition of moment of inertia, (1), p. 49, becomes



$$I = \int r^2 dA = \iint r^2 dxdy.$$

Hence by definition we have, from the figure,

XI $$\boldsymbol{I_x = \iint y^2 dxdy; \qquad I_y = \iint x^2 dxdy.}$$

The distance r in the general formula becomes y or x according as the moment of inertia is referred to the axis OX or OY.

The moment of inertia with respect to the origin O is

$$I_O = \iint \overline{OP}^2 dxdy = \iint (x^2 + y^2)\, dxdy.$$

Hence, adding equations **XI**, we obtain

XII $$\boldsymbol{I_O = I_x + I_y.}$$

Theorem. *The moment of inertia with respect to a point in a plane is equal to the sum of the moments of inertia with respect to two perpendicular axes lying in the plane and passing through the point.*

For a plane area equation **X**, p. 47, becomes

XIII $$\boldsymbol{I_l = I_{l'} + Aa^2.}$$

The proof is obvious and is left to the student as an exercise.

Polar coördinates. In polar coördinates the element of area is approximately $\Delta S = \rho\Delta\rho\Delta\theta$ (p. 5). Hence for polar moments of area we have, since $x = \rho\cos\theta$ and $y = \rho\sin\theta$ (figure, p. 50),

XIV
$$\begin{cases} I_x = \iint y^2 \cdot \rho d\rho d\theta = \iint \rho^3 \sin^2\theta\, d\rho d\theta, \\ I_y = \iint x^2 \cdot \rho d\rho d\theta = \iint \rho^3 \cos^2\theta\, d\rho d\theta. \end{cases}$$

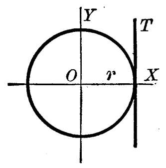
39. Calculation of moments of inertia of curved or space areas. For a curved area $\Delta S = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} \Delta x \Delta y$, and we define

$$I_{yz} = \iint x^2 dS,$$

with similar forms for I_{xy}, I_y, etc. Equations **II–X** apply also to curved areas if Δm is replaced by ΔS.

When thin plates or shells of the same form as the areas above are considered, the formulæ are exactly the same except that the density τ enters, and this may or may not be a constant, as has been explained before.

Illustrative example. *Given a circle $\rho = a$, to find moments of inertia.* Take first the moment of inertia with respect to a diameter. The integration may be performed by taking the whole circle, or by integrating over half and doubling it, or by integrating over one quadrant and multiplying by 4. Taking half the circle,



XV
$$I_y = 2\int_0^a \int_0^\pi \rho^3 \cos^2\theta\, d\theta d\rho = \frac{\pi a^4}{4} = \frac{A}{4} a^2.$$

Theorem. *The moment of inertia of a circle with respect to a diameter is equal to the square of the radius times one fourth the area.*

By means of relation **XIII** we find for the moment of inertia with respect to a tangent T,

$$I_T = \frac{Aa^2}{4} + Aa^2 = \frac{5}{4} Aa^2.$$

To find the moment of inertia with respect to a line through the center perpendicular to the plane of the circle, we may either integrate directly, as

$$I_O = \int_0^{2\pi} \int_0^a \rho^3 d\theta d\rho = \frac{A}{2} a^2,$$

or we may use equation **XII**, p. 50,

XVI
$$I_O = I_x + I_y = \frac{A}{4} a^2 + \frac{A}{4} a^2 = \frac{A}{2} a^2.$$

Theorem. *The moment of inertia of a circle with respect to an axis through its center perpendicular to its plane is equal to the square of the radius times one half the area.*

From the foregoing results we may obtain, by means of the theorem of parallel axes, the moment of inertia of the circle with respect to any line lying in its plane or perpendicular to its plane. For instance, consider the moment of inertia with respect to the line l perpendicular to the plane of the circle and passing through its circumference;

$$I_l = I_O + Aa^2 = \frac{A}{2}a^2 + Aa^2 = \frac{3}{2}Aa^2.$$

PROBLEMS

1. A straight rod of negligible mass and length a has five particles of equal mass situated on it at equal intervals of $\frac{1}{4}a$. Find I and r_0^2, (1) with respect to one end; (2) with respect to the middle point; (3) find I when the masses increase in arithmetical progression from the end.

2. Given three particles of equal mass situated at the vertices of an equilateral triangle. Find (1) I and r_0^2 with respect to one side; (2) with respect to a line parallel to one side passing through the opposite vertex.

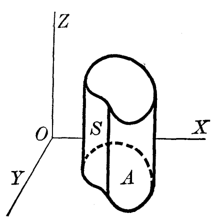
3. A regular hexagon has particles at the middle points of five of its sides. The masses of the particles taken in order are as 1, 2, 3, 4, 5. Find I and r_0^2 with respect to the unweighted side.

4. Find the moment of inertia of a rectangle (1) with respect to an axis through the center of area parallel to the longer side; (2) with respect to the longer side; (3) with respect to a line perpendicular to the plane at the center of one side; (4) with respect to a line perpendicular to the plane at one corner; (5) with respect to a diagonal.

5. Find I and r_0^2 for the area of an ellipse (1) relative to the axes; (2) relative to the origin; (3) relative to tangents at extremities of axes.

6. Find I for the area of an ellipse (1) relative to an axis perpendicular to the area and passing through the center; (2) relative to an axis perpendicular to the area through the focus; (3) relative to a plane through the minor axis perpendicular to the area; (4) relative to a tangent plane perpendicular to the major axis.

40. Relation between moments of inertia of areas and of right cylinders.* Let there be given any area A in the plane XY and suppose that upon this as a base a homogeneous right cylinder S is constructed, having a density τ and a height h. Then for the cylinder with respect to the plane YZ,



$$I_{yz} = \tau\iiint x^2 dz dy dx = \tau\int_0^h dz\left[\iint_A x^2 dx dy\right] = \tau h\iint_A x^2 dx dy.$$

* "Cylinder" is used in a general sense to include prisms, etc.

But by **XI**, p. 50, $\iint\limits_A x^2 dxdy$ is the moment of inertia of the base A with respect to the axis of Y. Hence

XVII $$\boldsymbol{I_{yz}\,(\text{of } S) = \tau h \cdot I_y\,(\text{of } A).}$$

Now let r_0 be the radius of inertia of the cylinder with respect to the plane YZ, and r_0' the radius of inertia of the base with respect to the axis OY. Then by **I**, p. 44, and (**5**), p. 50, **XVII** may be written

$$Mr_0^2 = \tau h A r_0'^2.$$

But $M = \tau h A$ and therefore $r_0^2 = r_0'^2$. Hence *the radii of inertia of any right cylinder and of its base with respect to a plane parallel to the elements of the cylinder are equal.*

Since for the cylinder $Mr_0^2 = I_{yz}$ and for the base $Ar_0'^2 = I_y$, we have the

Theorem. *If the moment of inertia of any plane area with respect to a line in its plane is known, then the moment of inertia of a homogeneous right cylinder of which the given area is the base, with respect to a plane through the line parallel to the elements of the cylinder, may be found by replacing the area in the known moment of inertia by the mass of the cylinder.*

If the moment of inertia of a plane area with respect to a point in its plane is known, the same process leads at once to the following

Theorem. *If the moment of inertia of any plane area with respect to a point lying in its plane is known, then the moment of inertia of a homogeneous right cylinder of which the given area is the base, with respect to a line through the point parallel to the elements of the cylinder, is found by replacing the area in the known moment of inertia by the mass of the cylinder.*

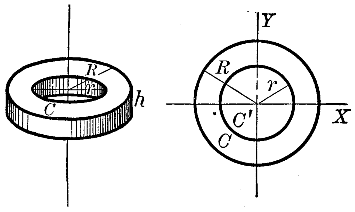
Illustrative example. *It is required to find the moment of inertia of a right circular cylinder* (1) *with respect to a plane through the axis;* (2) *with respect to the axis.*

By **XV**, p. 51, the moment of inertia of a circle of radius a (1) with respect to a diameter is $\frac{A}{4}a^2$; (2) with respect to its center is $\frac{A}{2}a^2$. Hence by the above theorems, replacing A by M, we have the following

Theorem. *The moment of inertia of a right circular cylinder* (1) *with respect to a plane through its axis is equal to the square of the radius times one fourth the mass;* (2) *with respect to its axis is equal to the square of the radius times one half the mass.*

41. Moments of inertia of combined solids and combined areas. If a solid is composed of two or more parts, the moment of inertia of the whole is evidently the same as the sum of the moments of inertia of its parts. Hence, if a portion be removed from a solid, the moment of inertia of what remains is equal to the moment of inertia of the whole *minus* the moment of inertia of the part removed. This principle applies without change to areas.

As an example consider the moment of inertia with respect to its axis of a hollow cylinder formed by cutting out of a right circular cylinder C of radius R, height h, and constant density τ, a concentric circular cylinder C' of radius r.



As we have just shown in the case of the solid cylinder, we may consider instead of the moment of inertia of the hollow cylinder the moment of inertia of its cross section in the plane XY with respect to its center O. Applying the theorem, p. 52, for the moment of inertia of a circle for an axis through its center perpendicular to its plane, we have for the circle C

$$I_O = \frac{AR^2}{2},$$

and for the circle C'

$$I_O = \frac{A'r^2}{2},$$

where A and A' denote the areas of C and C' respectively. Hence the moment of inertia of the ring with respect to O is

$$I_O \text{ for the ring} = \tfrac{1}{2}(AR^2 - A'r^2).$$

Since $A = \pi R^2$ and $A' = \pi r^2$, this may be written

$$I_O \text{ for the ring} = \frac{\pi}{2}(R^4 - r^4) = \frac{\pi}{2}(R^2 + r^2)(R^2 - r^2).$$

Denoting the area of the ring by A_0, we have $A_0 = A - A' = \pi R^2 - \pi r^2$, and hence

$$\boldsymbol{I_O \textbf{ for the ring} = \frac{A_0}{2}(R^2 + r^2).}$$

That is, *the moment of inertia of a plane area lying between two concentric circles of radii R and r, with respect to its center, equals the sum of the squares of the inner and outer radii times one half the area.*

Replacing A_0 by the mass of the hollow cylinder according to the theorem, p. 53, we obtain

$$\boldsymbol{I_O} \textbf{ for the cylinder} = \frac{\boldsymbol{M}}{\mathbf{2}}(\boldsymbol{R}^2 + \boldsymbol{r}^2).$$

Theorem. *The moment of inertia of a homogeneous hollow cylinder with respect to its axis equals the sum of the squares of the outer and inner radii times one half the mass.*

PROBLEMS

1. Find the moment of inertia of the hollow cylinder of p. 54 with respect to a line perpendicular to the XY-plane (1) through the outer circumference; (2) through the inner circumference.

2. Find the moment of inertia of the circular ring, p. 54, relative to OX.

Ans. $\frac{A_0}{4}(R^2 + r^2)$.

3. Find the moment of inertia of the hollow cylinder, p. 54, with respect to a plane through the axis.

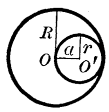
Ans. $\frac{M}{4}(R^2 + r^2)$.

4. Find the moment of inertia of the ring with respect to tangents to the circles C and C'.

5. Find the moment of inertia of a thin circular plate of uniform density (1) with respect to a diameter; (2) with respect to a tangent. *Ans.* (1) $\frac{1}{4}Ma^2$; (2) $\frac{5}{4}Ma^2$.

6. Find the moment of inertia of a right circular cylinder of radius a and height $2h$ with respect to its axis.

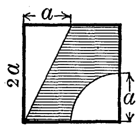
7. Find the moment of inertia of a circular area having a smaller circular area cut from it as in the figure, (1) with respect to a line through O perpendicular to the plane of the circle; (2) with respect to a diameter through O; (3) with respect to a line through O' perpendicular to the plane of the circle; (4) with respect to O and O'; (5) with respect to a line through O' parallel to a diameter through O. How will the results be changed if $2r > R$?



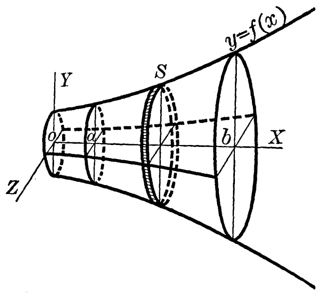
8. From a circle of radius R a rectangle is removed, the diagonals of the rectangle intersecting at the center of the circle. Find the moment of inertia (1) with respect to an axis perpendicular to the plane through one corner of the rectangle; (2) with respect to a diameter which is also a diagonal of the rectangle.

9. Find the moment of inertia of a homogeneous hexagonal plate with reference to a diagonal. *Ans.* $I = \frac{5}{24}a^2M$.

10. A homogeneous square plate whose side is $2a$ has parts removed as indicated in the figure. Find the moment of inertia with respect to the lower edge and with respect to an axis perpendicular to the plane of the plate at its center.



42. Moments of inertia of solids of revolution. The calculation of moments of inertia is greatly simplified in the case of solids of revolution. In some of the more important cases they may be found as follows.



With respect to the axis of revolution. Suppose $y=f(x)$ is the equation of the generating curve and consider the homogeneous solid generated by that portion of the curve included between the planes $x=a$ and $x=b$. Then by **VIII**, p. 47, $I_x=I_{xy}+I_{xz}$. In the present case, however, $I_{xy}=I_{xz}$ and hence

(1) $$I_x=2\,I_{xz}=2\,I_{xy}=2\,\tau\iiint y^2dzdydx \qquad \text{by III, p. 46}$$
$$=2\,\tau\int\left[\iint y^2dzdy\right]dx.$$

But $\iint y^2dzdy$ is the moment of inertia of a circular cross section S in which x is constant, with respect to a diameter. This equals the square of the radius y times one fourth the area πy^2, by the theorem, p. 51.

(2) $$\therefore\ I_x=2\,\tau\int_a^b\frac{\pi y^4}{4}\,dx=\frac{\pi\tau}{2}\int_a^b y^4dx,$$

which may be integrated when y is expressed in terms of x from the equation of the generating curve.

Since the mass $M=\pi\tau\int_a^b y^2dx$, we have finally

XVIII $$\boldsymbol{I_x=\frac{M}{2}\frac{\int_a^b y^4dx}{\int_a^b y^2dx}}.$$

It is instructive to regard the derivation of **XVIII** as follows. Consider the solid to be divided into thin circular plates each of thickness Δx, by planes perpendicular to OX. Then by the theorem, p. 53, the moment of inertia of one of these plates with respect to OX equals the square of its radius y times one half its mass $\frac{\tau\pi y^4\Delta x}{2}$. That is, approximately,

$$\Delta I_x=\tfrac{1}{2}\,\tau\pi y^4\Delta x \quad \text{and} \quad I_x=\underset{\Delta x=0}{\text{limit}}\,\Sigma\frac{\tau}{2}\,\pi y^4\Delta x=\frac{\pi\tau}{2}\int_a^b y^4dx,$$

which is the same as (2).

With respect to an axis perpendicular to and intersecting the axis of rotation. Taking the axis OY, we have

(3) $$I_y = I_{xy} + I_{yz}.$$ by **VIII**, p. 47

But $$I_{xy} = \frac{1}{2} I_x = \frac{\pi\tau}{4}\int_a^b y^4 dx,$$ by (1) and (2)

$$I_{yz} = \tau \iiint x^2 dz dy dx = \tau \int \left[\iint dz dy\right] x^2 dx$$
$$= \tau \int \pi y^2 x^2 dx \quad \text{since} \iint dz dy = \pi y^2.$$

XIX $$\therefore \boldsymbol{I_y = \pi\tau \int_a^b \left(x^2y^2 + \frac{y^4}{4}\right) dx},$$

in which y is known as a function of x from the equation of the given curve.

We may also regard the derivation of **XIX** as follows. Consider the solid to be cut into thin plates as before. The moment of inertia of one of these plates with respect to a diameter parallel to OY is approximately $\frac{1}{4}\pi y^4 \tau \Delta x$ by pp. 53 and 54. The moment of inertia of a single plate with respect to OY is found by the theorem of parallel axes, p. 47, and is approximately

$$\Delta I_y = \pi\tau\left(x^2y^2 + \frac{y^4}{4}\right)\Delta x.$$

Hence

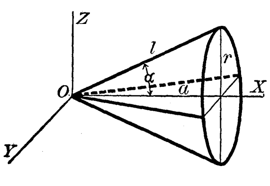
$$I_y = \lim_{\Delta x = 0} \Sigma \pi\tau \left(x^2y^2 + \frac{y^4}{4}\right)\Delta x = \pi\tau\int_a^b \left(x^2y^2 + \frac{y^4}{4}\right) dx$$

as before.

Equation **XIX** may be expressed in a form analogous to **XVIII**. Moments of inertia with respect to YZ, O, etc., are easily found by the aid of the relations derived on pp. 46 and 47.

Illustrative example. *Find the moments of inertia of a cone of height a formed by the revolution of the straight line $z = x \tan \alpha$ about the axis of X.* From equation (2), p. 56,

$$I_x = \frac{\pi\tau}{2}\int_0^a x^4 \tan^4 \alpha dx = \frac{\pi\tau a^5 \tan^4 \alpha}{10}.$$



Or, since $V = \frac{\pi a^3 \tan^2 \alpha}{3}$ and radius of base $= r = a \tan \alpha$,

(4) $$I_x = \frac{3M}{10} a^2 \tan^2 \alpha = \frac{3}{10} Mr^2.$$

This last result might have been obtained directly from equation **XVIII**.

From equation **XIX**,

$$I_z = \pi\tau \int_0^a \left(x^4 \tan^2 \alpha + \frac{x^4}{4} \tan^4 \alpha \right) dx$$

$$= \frac{\pi\tau a^5}{5} \left[\tan^2 \alpha + \frac{1}{4} \tan^4 \alpha \right] = \frac{\pi\tau a^5}{5} \left[\frac{r^2}{a^2} + \frac{r^4}{4\,a^4} \right]$$

(5) $$= \frac{\pi\tau a r^2}{20} (4\,a^2 + r^2) = \frac{3\,M}{20} (4\,a^2 + r^2).$$

By symmetry $I_y = I_z$.

43. Routh's rule. The following rule will be of assistance in remembering the moments of inertia of the elementary solids and areas, with respect to axes of symmetry.

The square of the radius of inertia with respect to an axis of symmetry is $\frac{1}{3}$, $\frac{1}{4}$, *or* $\frac{1}{5}$ *times the sum of the squares of the semi-axes perpendicular to the given axis according as the body is rectangular, elliptic, or ellipsoidal.*

By actual application it is found that this rule covers the cases of rectangles, parallelopipeds, spheres, cylinders, circles, ellipsoids, etc., all of whose volumes or areas are familiar, and from these data I is easily found. The rule may be illustrated as follows.

Consider a rectangle of sides $2\,a$ and $2\,b$. The square of the radius of inertia with respect to an axis through its center perpendicular to its plane is $r_0^2 = \frac{a^2 + b^2}{3}$, the semi-axes perpendicular to the given axis being a and b. The square of the radius of inertia with respect to an axis through the center of the rectangle perpendicular to the side $2\,a$ is $r_0^2 = \frac{a^2}{3}$, the only semi-axis perpendicular to the given axis being a.

An ellipse has semi-axes a and b. The square of the radius of inertia with respect to the major axis a is $r_0^2 = \frac{b^2}{4}$. With respect to an axis through its center perpendicular to its plane $r_0^2 = \frac{a^2 + b^2}{4}$. In the particular case of the circle of radius a, the radius of inertia with respect to every diameter becomes the same and is equal to $\frac{a^2}{4}$. For a circle with respect to an axis through its center perpendicular to its plane $r_0^2 = \frac{a^2 + a^2}{4} = \frac{a^2}{2}$.

An ellipsoid has semi-axes a, b, c. The square of the radius of inertia with respect to the axis $2c$ is $r_0^2 = \frac{a^2 + b^2}{5}$. In the particular case of the sphere, since the radii are all equal, evidently $r_0^2 = \frac{a^2 + a^2}{5} = \frac{2}{5}a^2$.

44. Products of inertia. The *product of inertia*, or **moment of deviation**, as it is sometimes called, is also a moment of the second order and is symbolized by $\int r_1 r_2 dm$, where r_1 and r_2 are the distances from some point in the element ΔM of a solid to *two* fixed planes; or, in the case of a plane area, r_1 and r_2 usually represent distances from a point in an element of area ΔA to *two fixed axes*, dA thus replacing dm in the integral. If we represent this integral by D, we may write

$$(1) \qquad D = \lim_{\Delta m = 0} \Sigma r_1 r_2 \Delta m = \int r_1 r_2 dm = \iiint r_1 r_2 \tau dz dy dx,$$

with a similar formula for areas.

Using a notation similar to that for moment of inertia, the product of inertia with respect to the XZ- and YZ-planes is

$$(2) \qquad D_{yz,\, xz} = \iiint \tau xy dx dy dz.$$

It will be found easy to write the various other integrals representing products of inertia with respect to the several planes and axes both in rectangular and in polar coördinates. The necessary transformations for solids and surfaces of revolution may very well be taken as an exercise.

It is easily seen that if either of the two planes or axes with respect to which the products of inertia are taken is a plane or an axis of symmetry, the products of inertia will vanish.

Theorems analogous to those for moment of inertia hold for products of inertia with respect to parallel axes and planes.

PROBLEMS

In the following problems results for moment of inertia alone are given. In most cases the radius of inertia may be determined by inspection. Additional examples may be obtained from the lists given with the two preceding chapters.

AREAS. ARCS

1. Given a uniform circular wire of outer radius R and inner radius r. Find the moment of inertia with respect to a diameter.

[Set $R = r + \Delta r$ and let Δr approach 0.] *Ans.* $\frac{Mr^2}{2}$.

2. Find the moment of inertia for a uniform wire in the form of an equilateral triangle of side a, (1) with respect to a line perpendicular to the plane of the triangle and equidistant from the vertices; (2) with respect to a line through a vertex perpendicular to the plane.

Ans. (1) $\frac{Ma^2}{2}$.

3. Find the moment of inertia of a thin plate bounded by the curve $y^2 = 4\,ax$ and by $x = a$, (1) with respect to the axis of X; (2) with respect to the origin, supposing the density to vary as y^2.

Ans. (1) $\dfrac{Ma^2}{5}$.

4. Find the moment of inertia of the area included by $x^2 - y^2 = 9$ and $x = 5$ (1) with respect to the axis of Y; (2) with respect to the focus.

Ans. (1) $410 - \frac{81}{4}\log 3$.

5. Find the moment of inertia for the arc of the parabola $y^2 = 4\,ax$ included between the origin and the point $(a,\ 2\,a)$, (1) with respect to the Y-axis; (2) with respect to the origin.

6. Find the moment of inertia for the area between the parabola $y^2 = 4\,ax$ and the line $y = 2\,x$, (1) with respect to the origin; (2) with respect to each of the axes.

Ans. (1) $\frac{33}{70}a^4$.

7. Find the moment of inertia of a loop of the lemniscate $\rho^2 = a^2 \cos 2\,\theta$ with respect to the origin; with respect to the Y-axis.

Ans. $\dfrac{\pi a^4}{16}$.

8. Find the moment of inertia for the arc of the cycloid $x = a(\theta - \sin\theta)$ $y = a(1 - \cos\theta)$, relative to the base.

Ans. $\frac{32}{15}a^3$.

9. An arc of the circle $\rho = a$ subtends an angle $2\,\theta$ at the center. Find the moment of inertia with respect to the middle of the arc. Apply to the special cases of a semi-circular arc and of a complete circle.

Ans. For the arc, $2\,a^2\tau s\left(1 - \dfrac{\sin\theta}{\theta}\right)$. For the semi-circular arc, $2\,a^2 s\tau\left(1 - \dfrac{2}{\pi}\right)$. For the circle, $2a^2\tau s$.

10. A bridge girder has a cross section in the form of the letter H made of three equal rectangles of sides a and b, $(a > b)$. Regarding the cross section as a thin lamina, find the moment of inertia with respect to a line through the center of mass of the horizontal rectangle and parallel to the edge a of the vertical rectangles.

Ans. $\dfrac{M}{36}(7\,a^2 + 12\,ab + 8\,b^2)$.

11. Find the moment of inertia of the area common to the parabolas $y^2 = 4\,ax$ and $x^2 = 4\,ay$ with respect to the axes, the density varying as x^2.

Ans. $I_y = I_x = \dfrac{4^5 2^3 a^6}{3^3}k$; $I_0 = \dfrac{4^5 2^4 a^6}{3^3}k$.

12. A thin plate has the form of a regular polygon of n sides of length $2\,l$. Find the moment of inertia relative to a line through its center perpendicular to its plane.

Ans. $\dfrac{Ml^2}{6}\left(1 - 3\cot^2\dfrac{\pi}{n}\right)$.

SOLIDS. SURFACES

1. Find the moment of inertia for a cube with respect to one corner. Find the moment of inertia for the surface of the cube with respect to an axis through the center parallel to an edge.

2. Find the moment of inertia for a right circular cone with respect to an axis through its center of mass perpendicular to its axis, supposing the density to vary as x^n. Take axes as in the figure, p. 57.

3. Find the moment of inertia for a circular cylinder of altitude h and radius a (1) with respect to a line in the plane of its base intersecting its axis perpendicularly; (2) with respect to a line perpendicular to its axis at the distance c from one end; (3) with respect to a plane through its center of mass perpendicular to its axis; (4) with respect to an element.

Ans. (1) $I = \frac{M}{12}(3a^2 + 4h^2)$; (2) $I = \frac{Ma^2}{4} + \frac{M}{3}(h^2 - 3hc + 3c^2)$; (4) $I = \frac{5}{4}Ma^2$.

4. Show that a theorem similar to that for parallel axes holds for two points, one of which is the center of mass.

5. Find the moment of inertia for the surface of a sphere (1) with respect to a plane through its center; (2) with respect to a diameter.

Ans. (1) $I = \frac{Sa^2}{3}$; (2) $I = \frac{2}{3}Sa^2$.

6. Find the moment of inertia for an ellipsoid of revolution (1) with respect to the axis of revolution; (2) with respect to a line perpendicular to the axis through the focus. *Ans.* (1) $\frac{2}{5}Ma^2$.

7. Find the moment of inertia for a paraboloid of revolution included between the planes $x = 0$ and $x = a$, the radius of the base being b, (1) with respect to the axis of revolution; (2) with respect to a line perpendicular to the axis at the vertex, assuming the equation of the generating curve is $y^2 = 4ax$.

Ans. (1) $\frac{\tau\pi ab^4}{6}$; (2) $\frac{\tau\pi ab^2}{12}(b^2 + 3a^2)$.

8. Given a hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$. Find the moment of inertia with respect to the axes for that part cut off by the planes $z = \pm 2$.

9. Find the moment of inertia with respect to the coördinate axes for the solid formed by removing from a right circular cone of height H and radius a, another cone of the same base and axis but of height $h < H$.

10. Find the moment of inertia for the elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to its axis.

11. A peg top is composed of a cone of height h and a hemispherical cap of radius a. From the point up to a distance b the material is three times as dense as the rest. Find the moment of inertia with respect to the axis of rotation.

Ans. $\frac{\tau\pi a^4}{10}\left(\frac{8}{3}a + h - b + \frac{9b^3}{h}\right)$, where $b = \frac{h}{3}$; $r_0^2 = \frac{27}{106}a^2$.

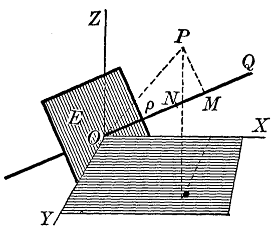
CHAPTER IV

ELLIPSOIDS OF INERTIA

45. Relation between moments of inertia relative to various lines and planes in space. A few instances have been discussed on pp. 45–47 in which moments of inertia with respect to different points, lines, and planes are not independent. The moment of inertia with respect to a point was seen to be related to the moment of inertia with respect to certain axes and planes through the point. The moment of inertia with respect to any plane was found to be connected with the moment of inertia relative to any parallel plane, and an analogous relation was observed to hold between two parallel axes. It follows, therefore, that if the moment of inertia for every line or plane through a given point is known, the moment of inertia for all lines or planes in space can be found. In order to find the moments of inertia for all lines or planes through a given point from simple data, let us consider the following problem.

Given the moments and products of inertia with respect to any three perpendicular lines (or planes) through a given point, to find the moments of inertia for all other lines (or planes) through the point.

Let OQ be any line for which the moments of inertia of a solid are required. Let P be any point whose coördinates are



x, y, and z. Draw PM perpendicular to OQ and let the direction cosines of OQ be α, β, γ. Also denote by A, B, C the moments of inertia of the solid relative to the axes X, Y, Z respectively; by A', B', C' the moments of inertia of the solid relative to the planes YZ, ZX, XY; by D, E, F the products of inertia (p. 59) relative to the planes YZ and XZ, YZ and XY, XZ and XY. Hence we have the following relations:

$$(1)\begin{cases} A = I_x = \int (y^2 + z^2)\,dm, & A' = I_{yz} = \int x^2 dm, & D = D_{yz,\,xz} = \int xydm, \\ B = I_y = \int (x^2 + z^2)\,dm, & B' = I_{xz} = \int y^2 dm, & E = D_{yz,\,xy} = \int xzdm, \\ C = I_z = \int (x^2 + y^2)\,dm, & C' = I_{xy} = \int z^2 dm, & F = D_{xz,\,xy} = \int yzdm. \end{cases}$$

From Analytic Geometry, p. 330, we have $\alpha^2 + \beta^2 + \gamma^2 = 1$, $\overline{OP}^2 = x^2 + y^2 + z^2$, $OM = x\alpha + y\beta + z\gamma$, and $\overline{PM}^2 = \overline{OP}^2 - \overline{OM}^2$.

Also $I_{OQ} = \int \overline{PM}^2 dm.$ Hence

$$\begin{aligned} I_{OQ} &= \int [(x^2 + y^2 + z^2) - (x\alpha + y\beta + z\gamma)^2]\,dm \\ &= \int [x^2(1-\alpha^2) + y^2(1-\beta^2) + z^2(1-\gamma^2) - 2\,\alpha\beta xy - 2\,\alpha\gamma xz - 2\beta\gamma yz]\,dm \\ &= \int [x^2(\beta^2 + \gamma^2) + y^2(\gamma^2 + \alpha^2) + z^2(\alpha^2 + \beta^2) - 2\,\alpha\beta xy - 2\,\alpha\gamma xz \\ &\qquad - 2\,\beta\gamma yz]\,dm \\ &= \alpha^2 \int (y^2 + z^2)\,dm + \beta^2 \int (x^2 + z^2)\,dm + \gamma^2 \int (x^2 + y^2)\,dm \\ &\qquad - 2\,\alpha\beta \int xydm - 2\,\alpha\gamma \int xzdm - 2\,\beta\gamma \int yzdm. \end{aligned}$$

$$\textbf{I}\quad \therefore\ \boldsymbol{I_{OQ} = \alpha^2 A + \beta^2 B + \gamma^2 C - 2\,\alpha\beta D - 2\,\alpha\gamma E - 2\,\beta\gamma F}$$
$$\boldsymbol{= I_x\alpha^2 + I_y\beta^2 + I_z\gamma^2 - 2\,D_{yz,\,xz}\,\alpha\beta - 2\,D_{yz,\,xy}\,\alpha\gamma - 2\,D_{xz,\,xy}\,\beta\gamma.}$$

When a system of particles is considered, the integral sign is replaced by Σ and dm by m. The significance of equation I may be expressed in the following

Theorem. *The moment of inertia with respect to any line through the origin is a homogeneous quadratic function of the direction cosines of the line.*

Again, if OQ is perpendicular to the plane E through O, we have for the moment of inertia with respect to E

$$\begin{aligned} I_E &= \int \overline{OM}^2 dm = \int (x\alpha + y\beta + z\gamma)^2 dm \\ &= \alpha^2 \int x^2 dm + \beta^2 \int y^2 dm + \gamma^2 \int z^2 dm + 2\,\alpha\beta \int xydm \\ &\qquad + 2\,\alpha\gamma \int xzdm + 2\,\beta\gamma \int yzdm. \end{aligned}$$

$$\textbf{II}\quad \therefore\ \boldsymbol{I_E = A'\alpha^2 + B'\beta^2 + C'\gamma^2 + 2\,D\alpha\beta + 2\,E\alpha\gamma + 2\,F\beta\gamma.}$$

Theorem. *The moment of inertia with respect to any plane through the origin is a homogeneous quadratic function of the direction cosines of the normal of the plane.*

46. Geometrical interpretation. The results of the preceding paragraph may be interpreted geometrically as follows. Let n be some arbitrary given constant, and lay off on OQ a length ON such that

$$\overline{ON}^2 = \frac{n^2}{I_{OQ}}. \tag{1}$$

That is, ON is laid off on OQ so that the square of ON is inversely proportional to I_{OQ}. Setting $ON = \rho$, we have

$$I_{OQ} = \frac{n^2}{\rho^2}. \tag{2}$$

If now for any given solid we lay off upon every radius vector from the origin a length ρ determined by equation (1), the locus of the point $N(xyz)$ is readily found to be a quadric. For the coördinates of N in any position are $x = \rho\alpha$, $y = \rho\beta$, $z = \rho\gamma$, and these being substituted in I give, with (2), the equation

$$Ax^2 + By^2 + Cz^2 - 2\,Dxy - 2\,Exz - 2\,Fyz = n^2, \tag{3}$$

which is a quadratic surface. Since by definition the moment of inertia is always positive (p. 44), every radius vector must be real, and hence the quadric denoted by (2) is an ellipsoid. This ellipsoid is called the **ellipsoid of inertia**, or the **momental ellipsoid for the point *O***. The axes of the ellipsoid are called the **principal axes for the point *O***, and the moments of inertia with respect to these axes, the **principal moments of inertia**.

When the axes of coördinates and the axes of the ellipsoid coincide, equation (2) reduces to

$$Ax^2 + By^2 + Cz^2 = n^2, \tag{4}$$

the ordinary equation of an ellipsoid. Hence in general, if there can be found three lines perpendicular to each other and intersecting in a point, such that if they be chosen as the axes of coördinates the products of inertia $\int xydm = \int yzdm = \int xzdm = 0$, then those lines are the principal axes of the body for that point.

The three planes, each of which is determined by two principal axes, are called the **principal planes** for the point.

If $z = 0$, that is, if the body is a plane area, the equation of the momental ellipsoid reduces to the equation of an ellipse called the **momental ellipse**, and this ellipse is the section of the momental ellipsoid corresponding to any point in the area, by the plane of the area.

At any point of a given material system there are always three principal axes. To prove this we have only to construct the ellipsoid of inertia corresponding to the point and then refer it to its principal diameters as axes, when the products of inertia all vanish.

A plane of symmetry of the given body is a principal plane of the ellipsoid of inertia (3) for every point in the plane. Hence, if XY is a plane of symmetry, $E = F = 0$ and OZ is a principal axis of the ellipsoid (3). If also YZ is a plane of symmetry, $D = 0$ and the ellipsoid is referred to its principal axes as in (4).

Example 1. *To find the moment of inertia of a homogeneous ellipsoid with reference to a diametral plane whose direction cosines with respect to the principal planes are α, β, γ.*

Let the diametral plane be denoted by E. Then from equation **II**, p. 63, the moment of inertia with respect to E is

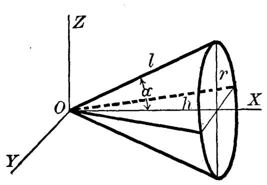
$$I_E = A'\alpha^2 + B'\beta^2 + C'\gamma^2 + 2\,\alpha\beta D + 2\,\alpha\gamma E + 2\,\beta\gamma F.$$

If we refer the equation of the ellipsoid to its principal diameters as axes, $D = E = F = 0$. Also from p. 48 we have

$$A' = \frac{Ma^2}{5}; \qquad B' = \frac{Mb^2}{5}; \qquad C' = \frac{Mc^2}{5}.$$

$$\therefore\ I_E = \frac{M}{5}(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2).$$

Example 2. *To find the moment of inertia of a homogeneous right cone of height h and radius r with respect to an element.* Choose the element l in the XZ-plane. Then by (1) we have



$$I_l = A\alpha^2 + B\beta^2 + C\gamma^2 - 2\,\alpha\beta D - 2\,\alpha\gamma E - 2\,\beta\gamma F.$$

But since the cone is symmetrical with respect to XY and XZ, all the products of inertia vanish; that is, $D = E = F = 0$. Also we have evidently

$$\alpha^2 = \frac{h^2}{h^2 + r^2}; \qquad \beta^2 = 0; \qquad \gamma^2 = \frac{b^2}{h_2 + r^2}.$$

and from previous results, p. 57,

$$A = \tfrac{3}{10}Mr^2; \qquad C = \tfrac{3}{20}M(r^2 + 4\,h^2) = B.$$

$$\therefore\ I_l = \frac{3}{10}Mr^2 \cdot \frac{h^2}{h^2 + r^2} + \frac{3}{20}M\,(r^2 + 4\,h^2) \cdot \frac{r^2}{h^2 + r^2} = \frac{3}{20}Mr^2\left(\frac{6\,h^2 + r^2}{h^2 + r^2}\right).$$

PROBLEMS

1. Find the moment of inertia of a rectangular plate of sides $2a$ and $2b$ with respect to a diagonal, and from this result deduce the corresponding result for a square.
Ans. $I = \frac{2}{3} M \frac{a^2b^2}{a^2 + b^2}$; $I = \frac{Ma^2}{3}$ for the square.

2. Find the moment of inertia for a diagonal of a parallelopiped whose edges are $2a$, $2b$, $2c$.
Ans. $I = \frac{2}{3} M \left(\frac{a^2b^2 + a^2c^2 + b^2c^2}{a^2 + b^2 + c^2}\right)$.

3. Find the moment of inertia of an ellipse for a diameter making an angle θ with the major axis.
Ans. $I = \frac{1}{4} M (a^2 \sin^2\theta + b^2 \cos^2\theta)$.

4. Find the moment of inertia for an ellipsoid with respect to a diameter whose direction cosines are l, m, n.

47. Properties of the principal axes. We have just shown that the ellipsoid of inertia corresponding to a given point was marked out by the motion of a radius vector which always preserved the relation (1), p. 64. From this relation and from the simple properties of ellipsoids the following theorems are seen to be true.

Theorem. *Of all the moments of inertia with respect to lines through a given point O, the greatest and the least are included in the principal moments of inertia.*

It is evident from (2), p. 64, that when ρ coincides with the longest semi-diameter I is least, and when ρ coincides with the shortest semi-diameter I is greatest.

A similar theorem holds with respect to the moments of inertia with reference to the principal planes passing through O.

Theorem. *If the three principal moments of inertia at any point O are equal, the ellipsoid of inertia corresponding to the point becomes a sphere.*

This conclusion follows at once from equation (4) if $A = B = C$.

In this case every diameter is a principal diameter, every line through O is a principal axis, and the moments of inertia with respect to all of them are equal.

For example, if the solid is a regular polyhedron, the ellipsoid of inertia for the center of the solid is a sphere.

Similarly for plane areas the momental ellipse corresponding to the center of any regular polygon becomes a circle.

Theorem. *If two of the principal moments of inertia with respect to any point O are equal, the momental ellipsoid corresponding to the point becomes an ellipsoid of revolution about the third principal axis.*

For in (4), p. 64, if $I_z = I_y$, then $B = C$, and (4) becomes the equation of an ellipsoid of revolution with respect to OX. The moments of inertia with respect to all lines in the YZ-plane are now equal.

PROBLEMS

1. Find the ellipsoid of inertia corresponding to the center of the homogeneous ellipsoid, p. 26.
Ans. $(b^2 + c^2)x^2 + (c^2 + a^2)y^2 + (a^2 + b^2)z^2 = \dfrac{5n^2}{M}.$

2. Given a homogeneous material straight line of length $2l$. Find the momental ellipsoid corresponding to a point O at a distance d from the center, O being the origin and the direction of the line the X-axis.
Ans. $z^2 + y^2 = n'^2$, where $n' = \dfrac{n}{M\left(\dfrac{l^2}{3} + d^2\right)}.$

3. Find the momental ellipsoid corresponding to the center of a material ellipse whose equation is $b^2x^2 + a^2y^2 = a^2b^2$.
Ans. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \left(\dfrac{a^2 + b^2}{a^2b^2}\right)z^2 = \dfrac{4n^2}{M}.$

4. Prove that the momental ellipse corresponding to the center of an equilateral triangle and to the center of a square is a circle. The same result holds for particles placed at the vertices of a similar weightless square or triangular frame.

5. Find the principal axes and the momental ellipsoid for one corner of a thin square plate, the area of the plate being in the XY-plane with a corner at the origin.
Ans. Momental ellipsoid, $2(x^2 + 2y^2 + z^2) - 3xz = \dfrac{6n^2}{Ma^2}.$

6. Using the result of problem 4, show that one momental ellipse corresponding to the center of a parallelogram is an ellipse touching the sides at their middle points.

7. Find position of the principal axes and the momental ellipsoid at the center of a homogeneous parallelopiped whose edges are $2a$, $2b$, $2c$. Apply results to a cube.
Ans. Principal axes are perpendicular to faces.
Momental ellipsoid, $(b^2 + c^2)x^2 + (c^2 + a^2)y^2 + (a^2 + b^2)z^2 = \dfrac{n^2}{M}.$
For a cube the momental ellipsoid is a sphere.

8. Find the principal axes and the ellipsoid of inertia corresponding to a corner of a cube.
Ans. Momental ellipsoid, $4(x^2 + y^2 + z^2) - 3(xy + xz + yz) = \dfrac{6n^2}{Ma^2}.$

9. Prove that the moment of inertia about all lines through the center of a cube is the same. Is this true of other regular solids?

10. Prove that if the height of a homogeneous right circular cylinder is to its diameter as $\sqrt{3} : 2$, the moments of inertia of the cylinder with respect to all axes passing through the center of mass will be equal.